

[Joke of the day: 64/16]

Understanding the logical implication: “ \rightarrow ”

Today we'll continue studying propositional logic. We're focused on the meaning of $p \rightarrow q$. It's subtler than you might think!

Let $p =$ “ x is a crow.”

Let $q =$ “ x is black.”

- (1) Now in groups, you decide which of these statements you think is true. For each that's true, mark it T. For each that's false, mark it F and give a counterexample: “ x might be a [something].” (We should agree for the sake of argument that crows are black. This is not an exercise in finding silly exceptions!)

$p \rightarrow q$	$q \rightarrow p$
$\neg p \rightarrow \neg q$	$\neg q \rightarrow \neg p$

- (2) How do you refute an implication? Suppose you wanted to prove that $p \rightarrow q$ was wrong. What would you have to do?

Understanding “ $p \rightarrow q$ ” when p is true is easy: q better be true!

Understanding “ $p \rightarrow q$ ” when p is false is complicated:

- (1) If it's raining, you'll want an umbrella: Certainly can agree, despite the fact that p is false!
- (2) If it's raining, it's a *dry* rain. (?!)
- (3) If it's raining, I'm the Dalai Lama. (???)

Normal English “if...then” sometimes invokes a hypothetical, forcing you to invent an alternate universe, then reason about it. This requires imagination and creativity, and cannot be done scientifically. Therefore we forbid hypothetical reasoning in mathematics.

“If $2 + 2 = 5$ then $x^2 = 47$.” **True**, on the grounds that it cannot be refuted, because refutation would demand *having* $2 + 2 = 5$. “There is nothing to worry about.”

“If $2 + 2 = 5$, then I am His Holiness the Dalai Lama.” **True** on the grounds that $2 + 2 \neq 5$.

Understanding the logical or: “ \vee ”

The sentence “Cats are mammals \vee water is wet” might seem objectionable because both are true. But in mathematics, we always accept the “or statement” as true when both parts are true. This is “*inclusive or*.” This distinguishes the technical mathematical “or” (\vee) from normal English “or,” which sometimes allows both (“milk or sugar?”) and sometimes disallows both (“coffee or tea?”).

Putting it all together: True or False? Answer in groups, and explain why. You'll be called to the board to explain your answers to the class!

(1) If $x^2 = -4$, then $x = \pi$.

(2) $(3 + 4 = 7) \vee$ (The Hausdorff-Besicovitch dimension of the Sierpinski Gasket is $\ln_2(3)$).

(3) $x \geq 5 \rightarrow (x \neq 0 \wedge x > 2)$.

(4) If x is a prime number, then 7 is a prime number.

(5) If you win the lottery, then you play the lottery.