

Math 340, Set Theory and Logic

Definitions

These are almost all of the definitions used in the course Math 340, Set Theory and Logic, with Jason Howald, Fall 2007. You will be required to gradually memorize this entire document. Don't panic – it's easier than it sounds. Italicized comments need not be memorized.

- (1) A mathematical theory is called **universal** if it is theoretically possible to understand all other mathematical questions in terms of that theory. *Finding a universal mathematical theory was a great achievement of early 20th century mathematics. Set theory is universal in the sense that all mathematical ideas can be boiled down to set theory. To express the universality of the theory, we must explain how things that do not look like sets, such as numbers, functions, and properties, can be understood as sets. There are several alternative universal theories.*
- (2) A **set** is a collection, grouping, or assembly of different things. A set cannot contain multiple copies of the same thing. *In a perfectly rigorous set theoretical system, the word “set” should be formally undefined, but I don't want to torture you with subtleties. Besides, even when we leave terms “formally undefined,” we all secretly know what they mean anyway.*
- (3) We write $x \in y$ to indicate that y is a set and x is one of its elements.
- (4) For sets A and B , we write $A = B$ to indicate that A and B have exactly the same elements. In other words, a set is completely determined by what it contains. *Unlike, say, a box, which has other properties in and of itself, such as size and color.*
- (5) For a set A , its **cardinality**, written $|A|$, is how many elements it has. *This definition isn't really fair if A is an infinite set. For that case, much more work is needed to understand cardinality.*
- (6) We write $\{e_1, \dots, e_n\}$, (called **list notation**) to indicate the set containing just those things listed.
- (7) We write $\{\text{[variable things]; [conditions]}\}$, (called **set builder notation**) to indicate all possible indicated things for which the indicated conditions are true.
- (8) We write \mathbb{Z} for the set of all integers (round numbers), including zero and the negatives.
- (9) We write ω for the set $\{0, 1, 2, \dots\}$ of all integers, including zero but excluding negatives.
- (10) We write \mathbb{N} for the set $\{0, 1, 2, \dots\}$, as another name for ω . *(Whether 0 is (should be?) an element of \mathbb{N} is debated. This question will not be a concern to us, but see Dr. Miller for details if you wish.)*
- (11) We write \mathbb{Q} for the set of all rational numbers, each of which can be written $\frac{p}{q}$, for integers p and q .
- (12) We write \mathbb{R} for the set of all real numbers.
- (13) For two sets A and B , their **Cartesian product**, written $A \times B$ is the set of all ordered pairs $\{(a, b); a \in A, b \in B\}$. *(“Cartesian” is an adjectival form of the last name of René Descartes, credited with the invention of the xy coordinate system, which can be understood as $\mathbb{R} \times \mathbb{R}$.)*
- (14) For two sets A and B , the **union** of A and B , written $A \cup B$, is the set $\{x; x \in A \text{ or } x \in B\}$. That is, all the elements of A with all the elements of B put together in one big set. *Mnemonics: the symbol \cup looks like a “U” for union, unite, unify, etc. In collective bargaining, a union is a large collective including people from different organizations.*
- (15) For two sets A and B , the **intersection** of A and B , written $A \cap B$, is the set $\{x; x \in A \text{ and } x \in B\}$. That is, the elements common to both sets. *(I don't know of any mnemonics for the symbol. “Intersect” is Latin inter- (between) + secare (to cut).)*
- (16) For two sets A and B , the **set difference**, written $A \setminus B$ or $A - B$ and pronounced “ A without B ” or “ A minus B ” is the set $\{x; x \in A \text{ and } x \notin B\}$. That is, the elements in A but not B .
- (17) For a set A , the **complement of A** , written A^c , is the set of everything *not* in A . *What, everything?? Including the mysterious motives of cats? This notation is used only when there is enough context to determine what kinds of things we're talking about, limiting the scope of “everything” appropriately.*
- (18) For two sets A and B , we say A is a **subset** of B , and write $A \subseteq B$, to mean $\forall x \in A, x \in B$.
Mathematicians seriously disagree about whether “ $A \subset B$ ” should allow for the possibility that $A = B$. (That is, if someone tells you that $A \subset B$, can you deduce that $A \neq B$?) Some hold that, by analogy with “ $x < y$ ” and “ $x \leq y$,” “ $A \subset B$ ” should not allow equality, and the symbol “ $A \subseteq B$ ” should be used if equality is allowed. Others prefer to write “ $A \subset B$ ” even in the case of equality, and reserve “ $A \subsetneq B$ ” to communicate that the sets are not equal. I was trained in the latter style, but the text uses the former style. I am attempting to change my habits and always write \subseteq for subsets, but if I should write $A \subset B$, it is more likely that I forgot than that I really meant $A \neq B$.

- (19) A **relation** R on two sets A and B is a subset of $A \times B$.
- (20) If D and R are sets, a **function** $f : D \rightarrow R$ is a relation on D and R with the property that $\forall x \in D \exists! y \in R (x, y) \in f$. For any given $x \in D$, we write $f(x)$ for that unique y for which $(x, y) \in f$. *This definition of a function makes a function a special kind of set, instead of a primary concept. Other systems make the function concept primary, but we have the universality of set theory in mind, so we have to understand everything as a set. We are rarely so committed to this function-as-a-set business that we will write peculiarities like $f \cap g$.*
- (21) For a function $f : D \rightarrow R$, the **domain of f** is the set D .
- (22) For a function $f : D \rightarrow R$, the **codomain of f** is the set R .
- (23) A valid **function definition** is a specification of the form “Let $f : D \rightarrow R$ via $f(x) = \dots$ ”, providing, in order, the function symbol (f), the domain (D), the range (R), and the action on a generic element x .
- (24) For a function $f : D \rightarrow R$, the **image of f** is the set $\{f(x); x \in D\}$, which need not be the entire set R .
- (25) For a function $f : D \rightarrow R$, the **range of f** is another word for the **image of f** . *Although the meanings of “codomain” and “image” are fixed and agreed upon, the meaning of “range” is in flux between its older usage (codomain) and its newer usage (image). Mathematicians use context to figure out what is intended, but learners should ask if unsure.*
- (26) For two functions $f : A \rightarrow B$ and $g : B \rightarrow C$, their **composition**, written $g \circ f$, is the function $\{(a, g(f(a))); a \in A\}$, with domain A and codomain C . Note: For $a \in A$, $(g \circ f)(a) = g(f(a)) \in C$. *If the functions are processes, make a bigger better process by doing one then the other. The symbol \circ can be pronounced “following” to keep track of which process is first. It can also be pronounced “composed with.” If the codomain of one doesn’t match the domain of the next, composition is impossible.*
- (27) A function $f : A \rightarrow B$ is called **injective** when $\forall x, y \in A$ if $f(x) = f(y)$ then $x = y$. In other words, no two domain elements go to the same range element. *Also called one-to-one, but this has deceptive connotations, so I prefer “injective.”*
- (28) A function $f : A \rightarrow B$ is called **surjective** when $\forall y \in B \exists x \in A f(x) = y$. In other words, everything in B is the image of something in A . *Also called “onto.”*
- (29) A function $f : A \rightarrow B$ is called **bijective** if it is both injective and surjective. A bijection creates a one to one correspondence between A and B .
- (30) Given a function $f : A \rightarrow B$, its inverse, written f^{-1} , is a function from B to A , so that for all $b \in B$, $f(f^{-1}(b)) = b$ and for all $a \in A$, $f^{-1}(f(a)) = a$. Not all functions have inverses, but if there is an inverse there is only one. *If there is an inverse for f , then f is bijective. This is usually the best way to prove that a function is bijective. the superscript $^{-1}$ is specialized notation and is not an exponent. There are no reciprocals here!*
- (31) Given a function $f : A \rightarrow B$ and a subset $Y \subset B$, the **preimage of Y** , written $f^{-1}(Y)$, is the set $\{x \in A; f(x) \in Y\}$, which is a subset of A . *Notice that the superscript $^{-1}$ is used in a different way here, and does not mean function inverse.*
- (32) Given a function $f : A \rightarrow B$ and a subset $X \subset A$, the **image of X** , written $f(X)$, is the set $\{f(x); x \in X\}$, which is a subset of B . *In the bizarre case that X might be both a subset of A and an element of A (e.g., $A = \{1, 2, 3, \{1, 2\}\}$), this notation is ambiguous and must be avoided.*