Brief Introduction to Calculus

Chapter 0

Introduction

Why study calculus? Anything that's managed to stay around for over 300 years might be worth a careful look. It has many direct applications. Calculus gives insight into how a falling body travels. The basic concepts of minimization and maximization, can be well understood through calculus. Calculus can also lead to understanding why soap bubbles are round. By the end of the semester, among other things, you should be able to show that among all rectangles with a fixed perimeter, the square encloses the largest amount of area. By the end of two semesters of calculus, you should be understand how to compute the volume of spheres, cones and other symmetric bodies. These are just a few of the many, many applications of calculus.

In addition to having many awesome applications, learning calculus will also help you improve your critical thinking and logical skills. Skills that would serve anyone regardless of their career plans.

We start with review. The topics we review will come up repeatedly, so they are worth studying.

0.1 The Real Numbers

The set of real numbers can be represented as the familiar number line:

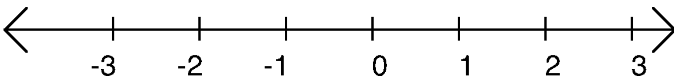

Figure 1: The familiar number line.

Note that as we move to the right on the number line, real numbers increase. We use the $>$ sign to indicate that one number is greater than (in other words to the right of) another. For example, $10 > 7$. It is also true that $-2 > -10$, since -2 lies to the right of -10. We can also say that $7 < 10$, in words “seven is less than ten.” We call any expression that involves greater than or less than an **inequality**. Here are some **rules for working with inequalities:**

1. If $a < b$, then $a + c < b + c$.
2. If $a < b$ and $c < d$, then $a + c < b + d$.
3. If $a < b$ and $c > 0$, then $ac < bc$.

   So far, these rules shouldn't surprise you. They work just like equations. The next two properties may catch you a little by surprise:

4. If $a < b$ and $c < 0$, then $ac > bc$. In other words, multiplying by a negative switches the inequality. For example, you probably agree that $2 < 3$, now multiply both sides by -2, flip and you get$(3)(-2) < (2)(-2)$, or $-6 < -4$, which is true!
5. If $0 < a < b$, then $1/a > 1/b$. This says that taking reciprocals of positive quantities will also flip your inequality. For example, again $2 < 3$ (still!), but $1/2 > 1/3$.

0.1.1 Subsets of the Real Numbers

The set of real numbers includes the **integers**, namely $0, \pm 1, \pm 2, \ldots$, the **rational numbers**, namely those numbers that can be expressed as a fraction p/q, where p and q are integers and q is not zero (remember: NEVER

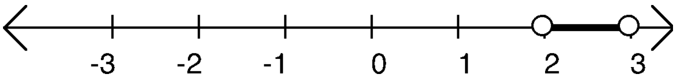

Figure 2: The interval $(2, 3)$.

divide by zero!). In fact, every real number can be written as a decimal. Those numbers that have a non-terminating, non-repeating expression are called **irrational numbers**. Perhaps the most famous irrational number is $\pi \approx 3.1415926\ldots..$, with $\sqrt{2} \approx 1.14\ldots$ another favorite. There are some difficult questions one can ask about the real numbers (don't worry, they won't be on the final): for example, are there "more" rational than irrational numbers? It is a fact that a "dart" thrown randomly at the number line will almost always hit an irrational. We'll save this interesting discussion for a course in Real Variables. In fact, Isaac Newton, one of the inventors of calculus, didn't even understand all of the properties of the real numbers. It wasn't for another 130 years after Newton's death that Richard Dedekind developed a proper definition of the real numbers.

We'll be particularly interested in studying interval of real numbers.

An **interval** is a subset of the real line that contains at least two numbers and it contains all of the real numbers lying between any two of its elements.

That's a fancy definition, let's look at some examples, so we can better understand what it's trying to say.

1. Example: The first interval we'll examine is the set of all real numbers greater than 2 and less than 3. So 2.4 lies in this set, but 1.9 does not, and neither does 2 and neither does 3, and neither does 3.1, and neither does 3.8, but 2.885 does. We'll denote this interval by $(2, 3)$, and we will also denote this set by $\{x : 2 < x < 3\}$. In words: "the set of all x such that 2 is less than x and x is less than 3." On the number line, we will indicate this interval by shading in, and using open circles at the endpoints. The open circles indicate that we do not want to include the endpoints. An interval that does not include its endpoints is called **open**. The interval $(2, 3)$ is open.

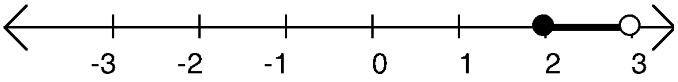

Figure 3: The interval $[2, 3)$.

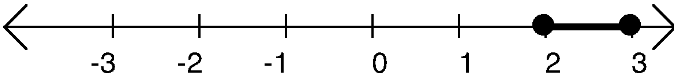

Figure 4: The interval $[2, 3]$.

2. Example: The interval $[2, 3)$ is nearly the same as the interval from the previous example, except that the number 2 is included in the set. In set builder notation, we denote this set by $\{x : 2 \leq x < 3\}$, "the set of all x such that 2 is less than or equal to x and x is less than 3." On the number line, we indicate this interval by shading, with a darkened circle at 2, and an open circle at 3. An interval that includes only one of its endpoints is called **half-open**. The interval $[2, 3)$ is half-open.

3. Example: The interval $(2, 3]$ is nearly the same as the interval from the previous example, except that the number 2 is not included in the set, and the number 3 is. In set builder notation, we denote this set by $\{x : 2 < x \leq 3\}$, "the set of all x such that 2 is less than x and x is less than or equal to 3." On the number line, we indicate this interval by shading, with an open circle at 2, and a darkened circle at 3. The interval $(2, 3]$ is half-open.

4. Example: This interval $[2, 3]$ includes all real numbers between 2 and 3, and includes 2 and 3. We also denote this by $\{x : 2 \leq x \leq 3\}$. "The set of all x such that 2 is less than or equal to x and x is less than or equal to 3." An interval that includes both of its endpoints is called **closed**. The interval $[2, 3]$ is closed.

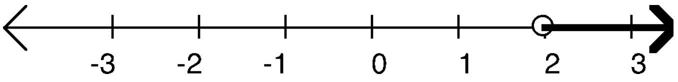

Figure 5: The interval $(2, \infty)$.

Examples involving infinity:

5. Example: The interval $(2, \infty) = \{x : 2 < x\}$ includes all real numbers larger than 2. We indicate this interval on the number line by drawing an open circle at 2, and shading all points to the right of 2. The interval $(2, \infty)$ is open.

6. Example: $[2, \infty) = \{x : x \geq 2\}$. This is a half open interval.

7. Example: $(-\infty, +\infty)$. This is the open interval consisting of all real numbers.

0.1.2 Solving Inequalities:

The idea behind solving an inequality is basically the same as the idea behind solving an equation. You want to isolate your unknown variable. Let's do an example.

Example: Solve the inequality: $-2x + 6 < 12$.

First subtract 6 from both sides to get $-2x < 6$. Then divide by -2 to get $x > -3$. So, provided that $x > -3$, we have that $-2x + 6 < 12$, and if $-2x + 6 < 12$, then $x > -3$. Try $x = 0$. Is $2(0) + 6 < 12$? Yes it is, since $6 < 12$.

0.1.3 Absolute Value

Here's another operation that you have seen before. The absolute value of a number is the distance from 0 to that number, and distance is always positive or 0. Intuitively speaking, the absolute value of a non-negative number gives back the same number, and the absolute value of a negative number gives a positive number of the same size. The more formal definition follows:

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

This definition works because the negative of a negative is positive. So, if $x < 0$ (x is negative, even though there is no negative sign written!), then $-x > 0$.

As usual, there are certain rules that the absolute value follows:

1. $|a| = |-a|$
2. $|ab| = |a||b|$
3. $\dfrac{|a|}{|b|} = \left|\dfrac{a}{b}\right|$ for $b \neq 0$
4. $|a+b| \leq |a| + |b|$ (the famous triangle inequality.)

If you take our course in advanced calculus, the triangle inequality will come up very often.

Before we go on to some examples, there are three other properties of absolute value that you will find useful:

Suppose $a > 0$, then:
5. $|x| = a$ if and only if $x = \pm a$
6. $|x| < a$ if and only if $-a < x < a$
7. $|x| > a$ if and only if $x > a$ or $x < -a$

Now let's try some examples:

1. Example: Solve $|3x + 3| = 12$.

   Once again, the object is to get x by itself. It's best to get rid of the absolute value sign, using property 5 from above. From that property, we know that any x that satisfies $|3x+3| = 12$ must also satisfy either $3x + 3 = 12$ or $3x + 3 = -12$. Let's put each case side by side:

$3x + 3 = 12$	$3x + 3 = -12$
$3x = 9$ (subtract 3 from both sides)	$3x = -15$ (subtract 3 here too)
$x = 3$ (divide by 3)	$x = -5$ (divide by 3 again).

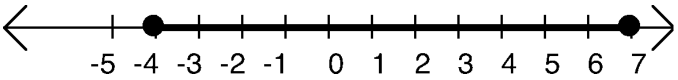

Figure 6: The solution set to $|2x-3|<11$.

So, either $x=3$ or $x=-5$ will satisfy the given inequality, so the solution is $x=3$ or $x=-5$. We might also say the solution set is $\{3,-5\}$.

2. Example: Solve $|2x-3|<11$

   In other words, we want to find all real values of x such that the distance from $2x-3$ to 0 on the number line is less than 11. Here we'll use property 6 of absolute value. We get a new inequality, without the absolute value sign:

   $$-11<2x-3<11$$

   Adding 3 to all three expressions involved, we get:

   $$-8<2x<14$$

   Now we'll divide all three terms by 2, on order to isolate x:

   $$-4<x<7$$

   And that's our solution. We can also denote our solution set by $\{x: -4<x<7\}$, and we can also sketch our solution on the number line (see figure).

3. Example: Solve $|2x-3|\geq 11$

   Here we also want to re-write our inequality, without the absolute value sign. Here we use the useful property 7 (with greater than or equal to

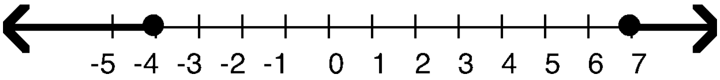

Figure 7: The solution set to $|2x-3| \geq 11$.

signs). Our original inequality can be broken up into the expressions $2x-3 \geq 11$ or $2x-3 \leq -11$. Let's solve these two inequalities:

$$\begin{array}{ll}
2x-3 \geq 11 & 2x-3 \leq -11 \\
2x \geq 14 \text{ (add 3 to both sides)} & 2x \leq -8 \text{ (add 3 here too)} \\
x \geq 7 \text{ (divide by 2)} & x \leq -4 \text{ (divide by 2 again).}
\end{array}$$

So, our desired solution set is $x \geq 7$ or $x \leq -4$. In set builder notation, that is $\{x : x \geq 7 \text{ or } x \leq -4\}$. In interval form, it's $(-\infty,-4] \cup [7,+\infty)$ On the number line, this looks like the sketch in figure 7.

You should look back at the previous two examples and observe the differences in technique used in each case. In particular, you should note that $|x| > a$ requires two inequalities, while $|x| < a$ requires just one (with three terms involved).

0.2 Coordinates and Lines

In calculus, we look at graphs and points in the two-dimensional plane. On the number line, a point corresponded to a single real number. Similarly, in the plane, a point will correspond to a single *pair* of real numbers. The person who came up with this correspondence was French philosopher René Descartes (1597-1650). As you will see, this will allow us to use algebra to describe geometric objects. For example, $y = x^2$ describes a parabola, $x^2 + y^2 = 1$ describes a circle. This is useful.

Let's describe the so-called Cartesian coordinate system. What is needed are two perpendicular axes (number lines), and an origin at the point where they intersect. By convention, we'll specify that the y-axis is vertical, and numbered from bottom to top, and that the x-axis is horizontal, and num-

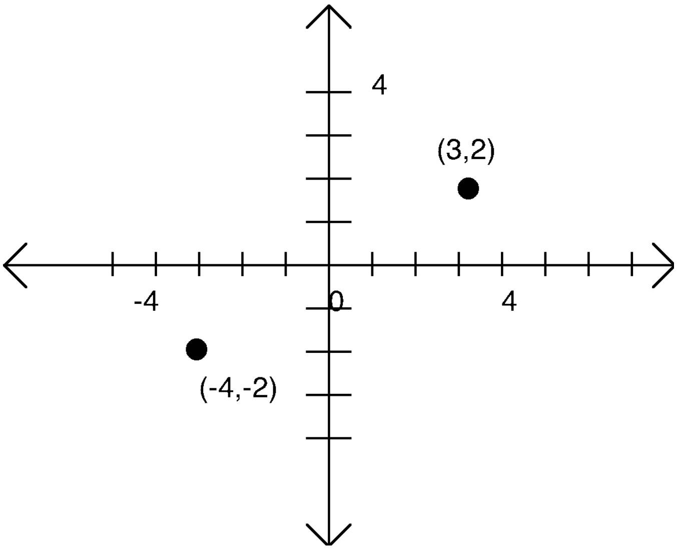

Figure 8: The cartesian plane.

bered from left to right. The origin will be given the coordinate $(0,0)$. In general, to plot the point (a,b), you would go exactly a units along the x-axis, then up b units.

0.2.1 Distance Between Two Points

Now we can easily calculate the distance between two points on the plane. Here's how. Consider two points $P = (x_1, y_1)$ and $Q = (x_2, y_2)$. For now, we'll assume that the line through P and Q is neither vertical nor horizontal. Using the line through P and Q, we can form a right triangle, with sides parallel to the coordinate axes. The distance from P to Q is the length of the hypotenuse of the triangle. The length of the horizontal side is $|x_1 - x_2|$, and the length of the vertical side is $|y_1 - y_2|$. Let the symbol d stand for the distance between P and Q, by the Pythagorean theorem, we can conclude that $d^2 = |x_1 - x_2|^2 + |y_1 - y_2|^2$. Taking square roots of both sides (don't worry about negatives since d is always positive), we get:

$$d = \sqrt{|x_1 - x_2|^2 + |y_1 - y_2|^2}$$

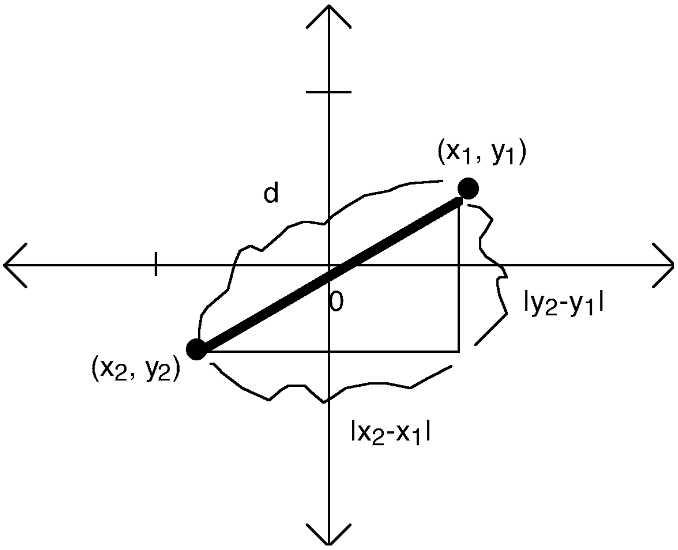

Figure 9: The distance between two points in the plane.

Since the square of a real number is always non-negative, we can omit the absolute value signs above, to get

the "Euclidean" distance formula between $P = (x_1, y_1)$ and $Q = (x_2, y_2)$:

$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

Does this work when the line through P and Q is horizontal? You bet, since in that case, $y_1 = y_2$, and thus $d = \sqrt{(x_1 - x_2)^2 + (0)^2} = \sqrt{(x_1 - x_2)^2} = |x_1 - x_2|$. The distance formula also works when the line through P and Q is vertical. Can you figure out why? If not, you should ask your instructor during office hours. Your instructor would be pleased to see you there.

Example:

Find the distance from the point $(1, 1)$ to $(4, 5)$.

Using the "Euclidean" distance formula, we get $d = \sqrt{(1 - 4)^2 + (1 - 5)^2} = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5$. That worked out nicely.

Exercise: Find the distance between the point $(1, 1)$ and $(1, 1)$ both using common sense, and using the distance formula.

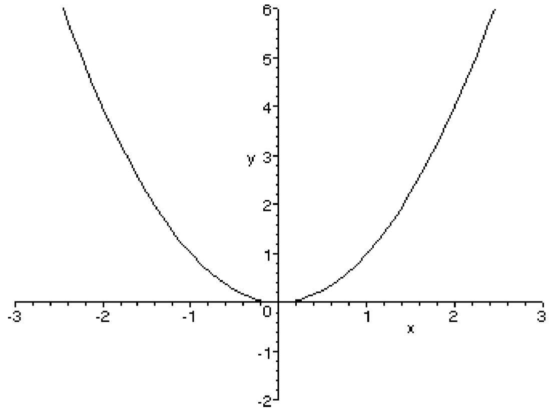

Figure 10: The graph of $y = x^2$.

0.2.2 Lines

Before we get to lines, we need the following definition:

The **graph** of an equation or inequality is the set of all points (x, y) whose coordinates satisfy the equation or inequality.

Example: Consider the graph of $y = x^2$. We can test whether or not certain points lie on the graph. For example, does $(2, 4)$ lie on the graph? We have to plug in $x = 2$ and $y = 4$ into our equation and then we have to check if the equation is still valid. In this case, we get $4 = 2^2$, which is valid. Is $(3, 3)$ on the graph? Certainly not, since $3 \neq 3^2$.

You may recall from previous experience that given two distinct point in the plane, there is a unique line through the two points. Next we'll introduce the concept of slope. This is an *extremely* important topic in calculus.

The **slope** of a line passing through the points (x_1, y_1) and (x_2, y_2) is given by: $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{\Delta y}{\Delta x} = \dfrac{rise}{run}$, where we must have $x_1 \neq x_2$.

Fact: for a given line this formula gives the same value, regardless of the two points selected. Remember that the slope of a vertical line is undefined (because we NEVER divide by zero), and the slope of a horizontal line is

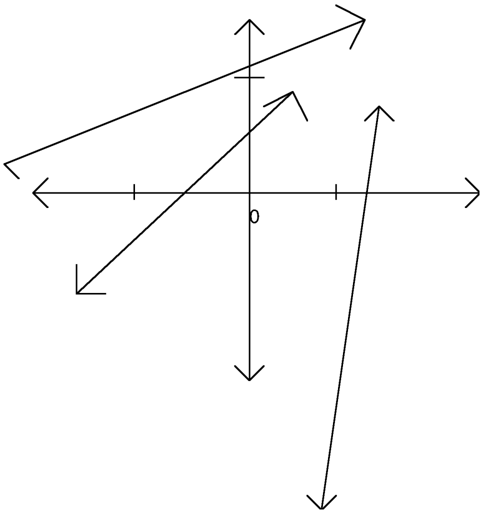

Figure 11: Some lines with positive slope.

0. This makes since, since you wouldn't want a horizontal line to have any slope, since it is as flat as possible. The slope tells the direction (uphill, downhill) and steepness of the line. A line with positive slope rises uphill to the right, and a line with negative slope falls downhill to the right. The larger the absolute value of the slope, the more rapid the rise or fall.

One the main topics you will study this semester is how to define slope for graphs that are not lines. This new version of slope will again give information about the steepness, rising, and falling of the graph.

0.2.3 Equations for Lines

There are two common forms of equations describing lines. One is the so-called

Point-Slope Form Equation of a Line: A line passing through the point (x_1, y_1) and having slope m is given by the equation $y - y_1 = m(x - x_1)$.

Example: Find the equation of the line through $(3, 0)$ and $(-1, 4)$.

First find the slope: $\frac{4-0}{-1-3} = -(4/4) = -1$. We'll use the point $(3, 0)$, although the other point would work just as well. Here's what the point-slope equation gives us:

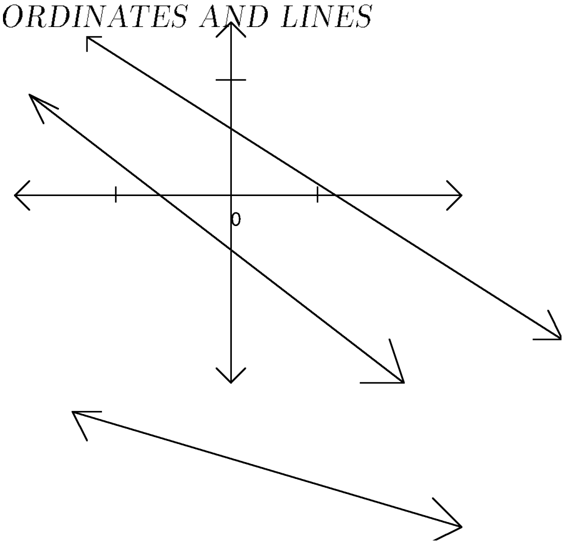

Figure 12: Some lines with negative slope.

$$y - 0 = (-1)(x - (3))$$

$$y = -x + 3$$

The other popular form for describing a line is the:

Slope-Intercept Form Equation of a Line:
An equation of the line with slope m and y-intercept b is given by the equation $y = mx + b$.

Remember that the y-intercept is the y-coordinate of the point at which the line crosses the y-axis. If you were given the line $y = -2x + 6$, you can tell right away that the slope of that line is -2 and the y intercept is 6.

Example: Use the slope-intercept form to find the equation of the line through the points $(3, 0)$ and $(-1, 4)$.

First find the slope. This was found earlier to be -1. So our equation is of the form $y = -x + b$. To find b, plug one of the two points into the equation. This time, we'll use $(-1, 4)$. Plugging in, we get: $4 = -1(-1) + b$, or $4 = b + 1$, or $b = 3$. So there it is again, the equation is $y = -x + 3$.

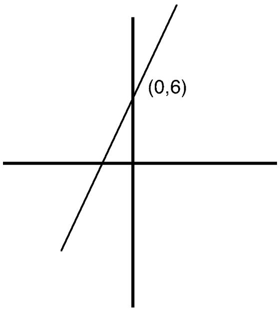

Figure 13: The y-intercept is 6.

0.3 Trigonometry

Our life on this earth is full of repeating phenomena. Our life on this earth is full of repeating phenomena. Every day, the sun comes up, and every evening the sun sets. We have four seasons that change at regular intervals. Trigonometry gives us the basic mathematical tools necessary for studying repeating phenomena. The word trigonometry is comprised of the two ancient Greek words ‘trigono’ or triangle and ‘metro’ or measure. We’ll start by studying angles, and talk about triangles later.

Most of the angles we study will be in so-called ”standard position”, that is to say, their vertex will be on the origin, and their initial side will lie on the positive $x-$axis. Positive angles move counter-clockwise, and negative angles move clock-wise. Another important object for us will be the unit (radius is 1) circle centered at the origin. The equation whose graph gives this circle is $x^2 + y^2 = 1$. We measure angles in radians. The **radian** measure of an angle θ equals the length of the arc subtended (determined) by θ on the unit circle.

You might recall that the circumference of the unit circle is 2π. Thus, an angle of 2π radians will cover the entire unit circle once. In other words, there are 2π radians in 360 degrees. We use the factor $\frac{2\pi}{360}$, or more simply $\frac{\pi radians}{180 degrees}$ to convert from degrees to radians. To convert from radians to

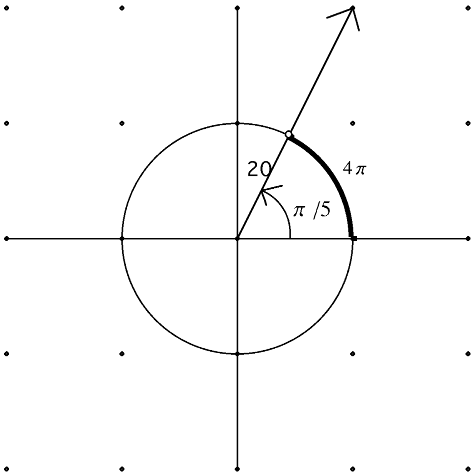

Figure 14: Radian measure.

degrees, use $\dfrac{180 degrees}{\pi radians}$.

<u>Example</u> Convert $\pi/4$ to degrees.

Multiply $\pi/4$ by $\dfrac{180 degrees}{\pi radians}$ to get $\dfrac{180\pi}{4\pi}$. Cancel common terms to get 45 degrees.

In calculus, it turns out that radians are much, much easier to deal with than degrees. This means it's time for you to move on from the more comfortable degrees, to the more useful radians. Be sure to keep your calculator in radian mode at all times. Here's a table to help you feel more comfortable with radians.

degree measure	radian measure
0	0
30	$\pi/6$
45	$\pi/4$
60	$\pi/3$
90	$\pi/2$
180	π
270	$3\pi/2$
360	2π

0.3.1 A nice formula: $a = r\theta$

Here's the scene: you have a circle of radius r, and an angle of θ. This angle θ subtends (good word, isn't it?) an arc of length a on the circle. It's a fact that the length of an arc is proportional to the size of the corresponding angle. We know the entire circle has an angle of 2π, and the corresponding arc (the entire circle) has length $2\pi r$ (=the circumference of the circle). We can thus set up the following proportion:

$$\frac{2\pi r}{2\pi} = \frac{a}{\theta}$$

We can cross-multiply to get rid of the fractions; the following equation results: $2\pi r\theta = 2\pi a$. Canceling the 2π out from both sides, we get $a = r\theta$.

Example: Given an angle of $\theta = \pi/5$, and a circle of radius 20, find the length of the arc subtended by θ on this circle.

Use $a = r\theta$. We get $a = 20(\pi/5) = 4\pi$. That's it, 4π is our answer.

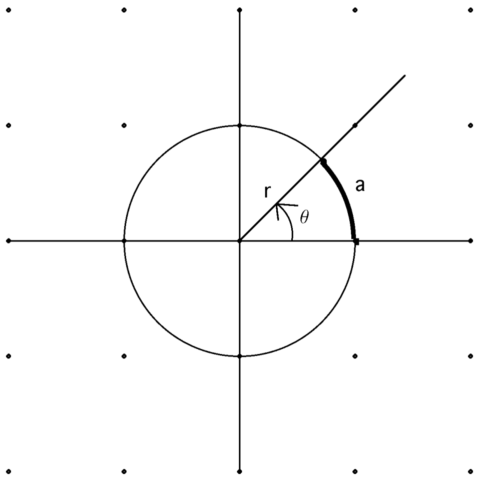

Figure 15:

0.3.2 Without Any Further Ado, the Trig Functions Themselves

We're about to discuss the trig functions including $sin(\theta)$ and $cos(\theta)$. The next section of these notes will talk about what a function is in a careful way. For now, what you need to know is that $sin(\theta)$ is like a machine that takes in an angle, namely θ, and spits out a number, namely $sin(\theta)$. The $cos(\theta)$ function works much the same way. Never forget that $sin(\theta)$ is not sin multiplied by θ. In fact, sin by itself has no meaning. To have any meaning at all, sin needs θ. Now the time is right to tell you exactly how to define the function $sin(\theta)$, and the other trig functions.

Picture an angle in standard position, and imagine the unit circle centered at the origin. You get a picture as in figure 1.15.

That angle intersects the unit circle at a point (x, y). That point, (x, y) depends directly on the value of θ. For example, if θ were 0, then the corresponding y value would be 0 (see figure).

If θ takes on a value of $\pi/2$, then the corresponding x value is 0. Given an angle θ, with corresponding point (x, y) on the **unit circle**, we define all of the trig functions as follows:

$sin(\theta) = y$	$csc(\theta) = 1/y$
$cox(\theta) = x$	$sec(\theta) = 1/x$
$tan(\theta) = y/x$	$cot(\theta) = x/y$

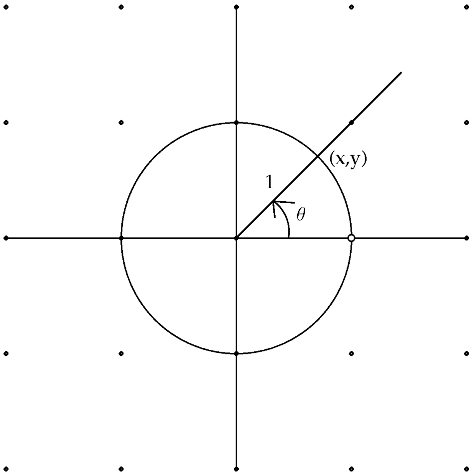

Figure 16: The unit circle, studied and loved by calculus students throughout the world.

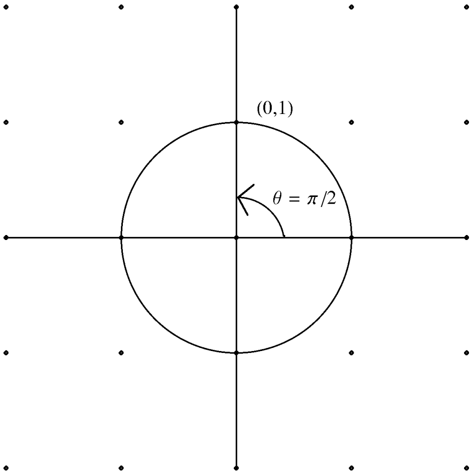

Figure 17: Here's why $cos(\pi/2) = 0$ and $sin(\pi/2) = 1$. The mystery is revealed!

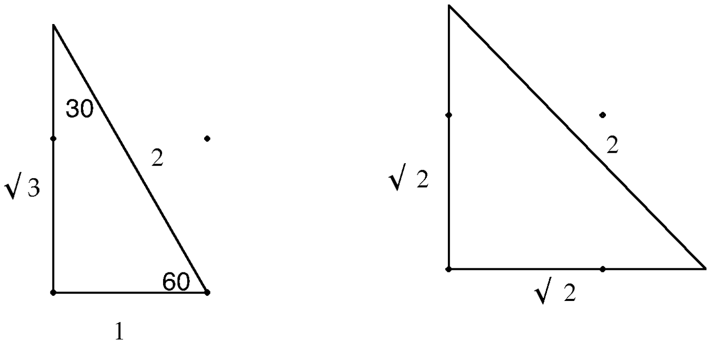

Figure 18: Look familiar?

Before we go on to examples, we should recall some famous triangle ratios. In particular, the 30-60-90 triangle, and the 45-45-90 triangle (here we use degrees only to jog your memory):

Example: Use the unit circle to determine the trigonometric ratios for $\theta = \pi/4$.

The ray emanating from the origin at an angle of $\theta = \pi/4$ will intersect the unit circle at the point $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.

Thus, using the 45-45-90 triangle ratios:

$$\begin{array}{ll} sin(\pi/4) = \frac{1}{\sqrt{2}} & csc(\pi/4) = \sqrt{2} \\ cos(\pi/4) = \frac{1}{\sqrt{2}} & sec(\pi/4) = \sqrt{2} \\ tan(\pi/4) = 1 & cot(\pi/4) = 1 \end{array}$$

Example: Use the unit circle to determine the trigonometric ratios for $\theta = 3\pi/4$.

This time, the ray emanating from the origin at an angle of $3\pi/4$ intersects the unit circle at the point $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$. (The x-coordinate is negative, since the point lies in the second quadrant.)

Again, using the 45-45-90 triangle ratios:

$$\begin{array}{ll} sin(\pi/4) = \frac{1}{\sqrt{2}} & csc(\pi/4) = \sqrt{2} \\ cos(\pi/4) = -\frac{1}{\sqrt{2}} & sec(\pi/4) = -\sqrt{2} \\ tan(\pi/4) = -1 & cot(\pi/4) = -1 \end{array}$$

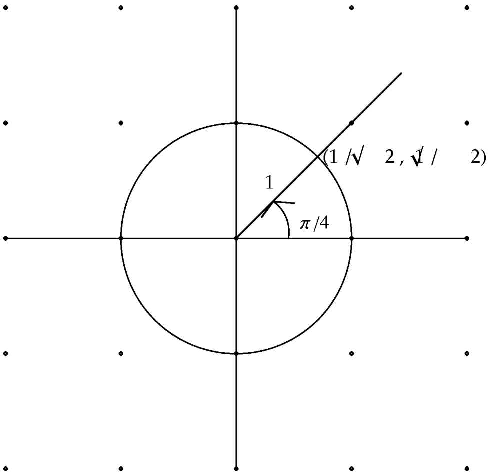

Figure 19: The unit circle used to determine the trigonometric ratios for $\theta = \pi/4$.

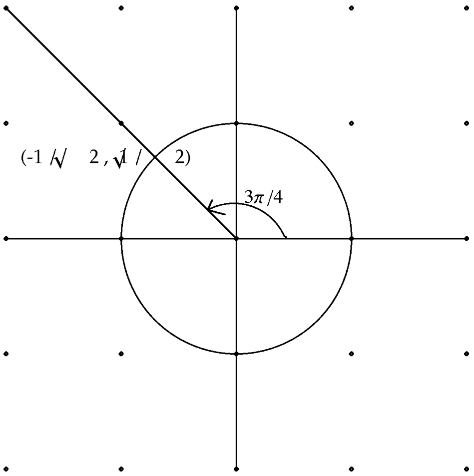

Figure 20: The unit circle used to determine the trigonometric ratios for $\theta = 3\pi/4$.

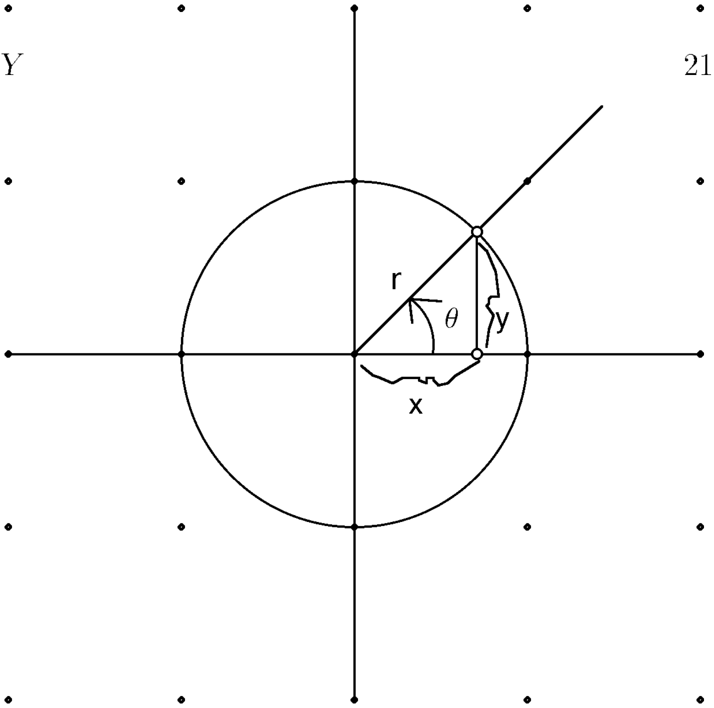

Figure 21: The connection between high school trig definitions, and circle based definitions of trig functions.

Some books define **the trig functions** for an angle **on the circle of radius r**. We'll do that here: Given an angle θ, with corresponding point (x, y) on the **circle of radius r**, we define all of the trig functions as follows:

$sin(\theta) = y/r$	$csc(\theta) = r/y$
$cox(\theta) = x/r$	$sec(\theta) = r/x$
$tan(\theta) = y/x$	$cot(\theta) = x/y$

This definition doesn't change anything. You probably remember the following from high-school:

$sin(\theta)$ =opp/hyp	$csc(\theta)$ =hyp/opp
$cox(\theta)$ =adj/hyp	$sec(\theta)$ =hyp/adj
$tan(\theta)$ =opp/adj	$cot(\theta)$ =adj/opp

In the table involving x, y and r, x plays the role of the adjacent side length, y plays the role of the opposite side length, and r plays the role of the length of the hypotenuse.

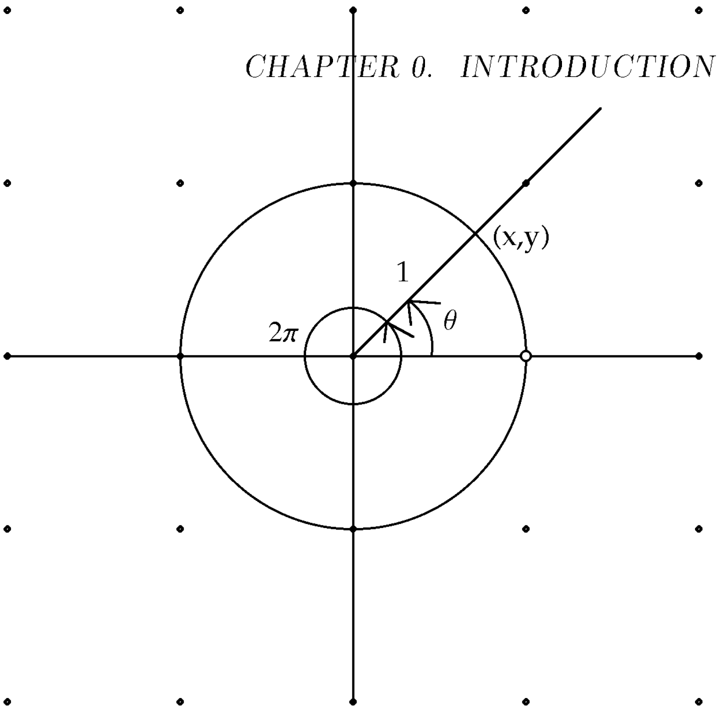

Figure 22: Going around an angle of 2π brings you back to where you started.

0.4 Graphs of Trig Functions

A periodic function is one that repeats its output values over and over as the input values vary. The horizontal length of the shortest pattern that repeats is called the period. As you go around the unit circle by an angle of 2π, you come back to the same point. This suggests that $y = \cos x$ and $y = \sin x$ are periodic with period 2π. (It turns out that $y = \tan x$ is periodic with period π.) This means that for any value of x, the following holds: $\sin(x + 2\pi) = \sin(x)$, and $\cos(x + 2\pi) = \cos(x)$.

Coming up is the graphs of $y = \sin x$. You should check that $(0, 0)$ lies on the graph (since $sin(0) = 0$), and similarly that $(\pi/2, 1)$ also lies on the graph, in addition to some points of your own choosing.

Also coming up is the graph of $y = \cos x$. Check that $(0, 1)$ is on the graph, in addition to some other familiar points.

As you may imagine, the graph of $y = \sec x = 1/\cos x$ is closely related to the graph of $y = \cos x$. Of course, $\sec x$ is undefined for those values of x that make $\cos x = 0$. In addition, $secx = cosx$ when $1/\cos x = cosx$, in other words, when $\cos^2 x = 1$ or when $\cos x = \pm 1$. This holds true for $x = 0, \pm\pi, \pm 2\pi, \ldots$. We graph $y = \cos x$ and $y = \sec x$ below.

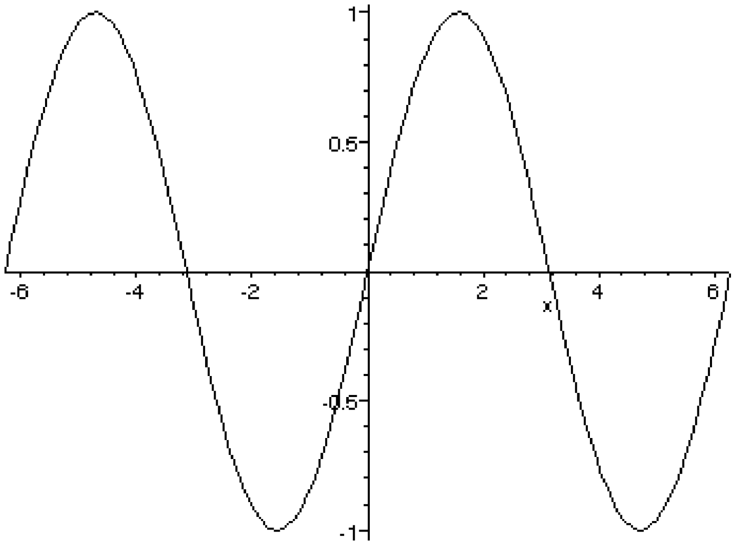

Figure 23: The graph of $y = \sin x$.

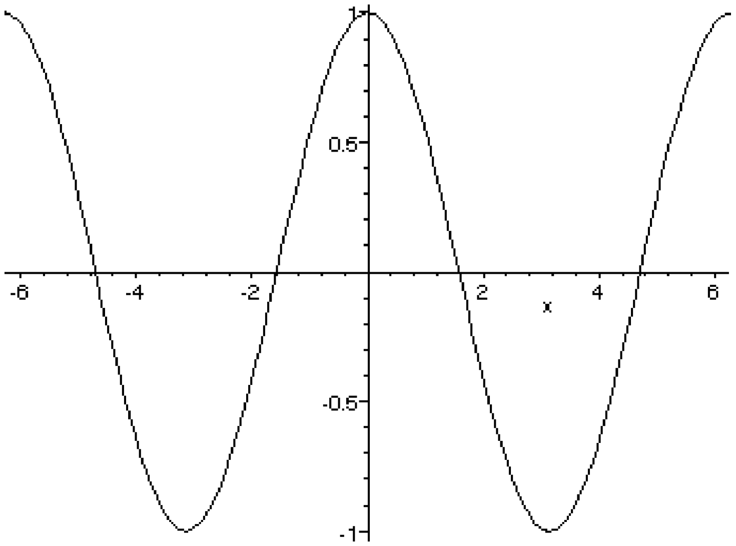

Figure 24: The graph of $y = \cos x$.

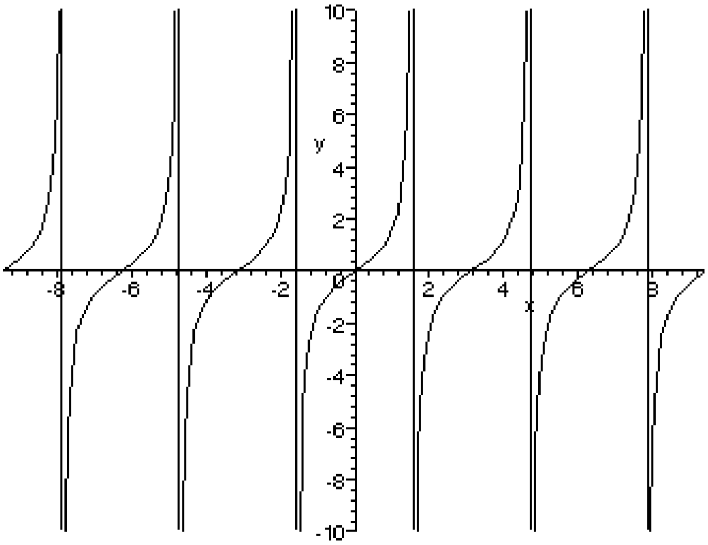

Figure 25: The graph of $y = \tan x$.

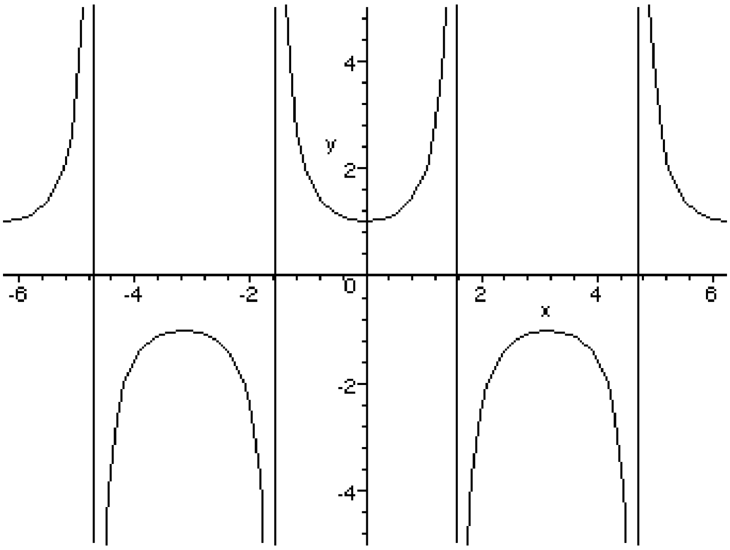

Figure 26: The graphs of $y = \cos x$ and $y = \sec x$.

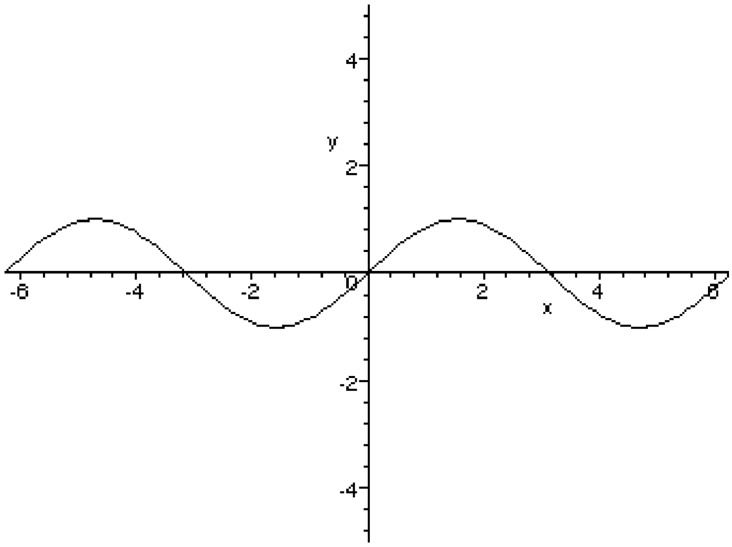

Figure 27: The graphs of $y = \sin x$ and $y = cscx$. They are similar in nature to the graphs of $y = \cos x$ and $y = \sec x$.