Graphs that are intrinsically linked with an unused vertex

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Abstract

We examine graphs that, for every spatial embedding, have a pair of linked cycles such that linked cycles do not pass through every vertex of the graph (we call such a graph *intrinsically linked with an unused vertex*). We demonstrate three graph operations that transform an intrinsically linked graph to a graph with this property. It remains an open question to determine all minor-minimal graphs that are intrinsically linked with an unused vertex.

1 Introduction

It has previously been shown by Conway-Gordon [2] and Sachs [10] that K_6 is *intrinsically linked*, that is, every spatial embedding of K_6 contains at least one pair of disjoint cycles that form a non-splittable link. Robertson, Seymour and Thomas [8] later proved that a graph is intrinsically linked if and only if it contains as a minor one of the seven graphs in the so called Petersen family (graphs that can be obtained from K_6 through Δ-Y exchanges and Y-Δ exchanges). The main focus of this paper is on the search for an analogue of this theorem for graphs that, for every spatial embedding, have linked cycles such that the linked cycles do not pass through every vertex in the graph. We call such a graph *intrinsically linked with an unused vertex*. Of course, any graph that consists of an intrinsically linked graph together with another graph component (such as an isolated vertex) will have this property. As the main result of this paper, we exhibit three non-trivial graph operations that transform an intrinsically linked graph to a graph that is intrinsically linked with an unused vertex.

All graphs considered shall be *simple* graphs. A graph is simple if there is at most one edge between any two verticies and there are no loops, that is, there are no edges connecting a vertex to itself. A *vertex splitting* of a vertex v in a graph G is achieved by replacing v with two verticies v' and v'', adding the edge $\overline{v'v''}$ and connecting a subset of the edges that were incident to v to v' and the rest of the edges that were incident to v to v''. A graph G is considered to be an *expansion* of a graph H if G can be obtained by vertex splittings of H. A *contraction* of an edge is the inverse of a splitting. If we contract the graph G along the edge e, we denote the resulting graph $G\backslash e$. A graph H is said to be a *minor* of a graph G if H can be obtained from G by deleting and/or contracting a finite number of edges, and possibly by removing isolated verticies. A graph G is *minor-minimal* with respect to a given property if and only if no minor of G has the property. It is well known (see [7], [6]) that if G is intrinsically linked and H contains G as a minor, then H is also intrinsically linked. The same reasoning also applies to graphs that are intrinsically linked with an unused vertex (see the next section for details). By the important result of Robertson and Seymour [9], there are only finitely many minor-minimal graphs that are intrinsically linked with an unused vertex. It remains an open question to determine all minor-minimal intrinsically linked graphs with an unused vertex. Our works suggests that the number of minor-minimal intrinsically linked graphs with an unused vertex is large.

A motivation in looking for graphs that are intrinsically linked with an unused vertex comes from trying to understand graphs with the disjoint linking property. A graph has the *disjoint linking property* if in every spatial embedding there exists a pair of non-splittable links that share no verticies and no edges. Of course, any graph that contains two disjoint sub-graphs that are each intrinsically linked will have this property. In [1] several non-trivial graphs with the disjoint linking property were given, and all of them were constructed using graphs that were intrinsically linked with an unused vertex. For example of the constructions they did, consider the 7 vertex graph $K_{3,1,1,1,1}$ which has a pair of linked triangles in every embedding, so has an unused vertex. By connecting all of the verticies of a K_5 to all of the verticies in the partition containing 3 elements, one creates a graph with the disjoint linking property. In a spatial embedding of this graph, the $K_{3,1,1,1,1}$ has a pair of linked triangles, and the unused vertex and the K_5 form an embedded K_6, thus also have a pair of linked triangles ([2], [10]). Determining all minor-minimal intrinsically linked with an unused vertex graphs seems difficult, but the problem of determining all minor-minimal graphs with the disjoint linking property is even more difficult.

Briefly, we describe our three non-trivial graph transformations here. The first non-trivial move, *triangle addition*, is the addition of a triangle along three mutually nonadjacent verticies. We restrict this move in that the three verticies cannot be the endpoints of a Y. We impose this restriction as all graphs in the Petersen family are obtained from $Y-\Delta$ and $\Delta-Y$ from K_6, so adding a triangle to a Y in a Petersen graph would result in a Petersen graph with an extra vertex (connected to the graph by three extra edges).

The second type of move, *triangle expansion*, involves choosing a single vertex and expanding it twice, and then connecting the endpoints of the expansion by an edge. Equivalently, one can think of this operation as replacing a vertex with a triangle, and then associating the old edges with the new verticies of the triangle in an arbitrary fashion. Here we impose the restriction that each vertex of the new triangle has at least one old edge incident to it. Otherwise, such a triangle expansion performed on a intrinsically linked graph would achieve the same result as expanding a vertex, then adjoining another vertex to the graph (connected by two new edges), which is a trivial way to obtain a graph that is intrinsically linked with an unused vertex.

The third move, *vertex addition*, is merely the addition of degree two verticies along existing edges. By adding degree 2 vertices to enough edges of an intrinsically linked graph, one can eventually create a graph that is intrinsically linked with an unused vertex.

Throughout the paper we will take an embedded graph to mean a graph embedded in 3-space, where all of our embeddings are tame. A *link* is a finite set of disjoint cycles embedded in 3-space. A link is said to be *non-splittable* if there exists no sphere in 3-space disjoint from the link that separates components of the link. We will use the terms *non-splittably linked cycles* and *linked cycles* interchangeably; both will represent cycles in a spatially embedded graph that form a non-splittable link. All linking numbers in this paper will be considered mod 2.

2 Proof of minor-closure

Here we show that if a graph G is not intrinsically linked with an unused vertex, and H is a minor of G, then H is also not intrinsically linked with an unused vertex. The contrapositive of this result follows from the next result. Our proof is based on the proof that vertex splitting preserves intrinsic linking presented in [5].

Proposition 2.1. *If G is intrinsically linked with an unused vertex, and G' is a graph obtained from G by a vertex splitting, then G' is also intrinsically linked with an unused vertex.*

Proof. Let G' be obtained from G by splitting the vertex v into the edge (v_1, v_2). Embed G'. Deform the embedding of G' to an equivalent embedding, so that there is a projection of the embedding with the edge (v_1, v_2) not crossed. Now we compare this specific projection of G' to the projection of G resulting from contracting (v_1, v_2) back to v. If the associated embedding of G contains a non-split link that does not pass through v, then the embedding of G' will also contain a non-split link that does not pass through v_1 (or v_2). We may thus assume that the embedding of G only contains non-split links that pass through the vertex v. Consider a non-split link that passes through v, and has a unused vertex, w. Let L be the set of all components of the link that do not contain v, and let M be the component that passes through v. Any component in L exists unaltered in the associated projection of the embedding of G', and let M' be M with the vertex v expanded to the edge (v_1, v_2). It follows that the components of L must still be linked with M' because no new crossings have been introduced to any component, and none have been eliminated. Moreover, the vertex w is not used in the link. It follows that the embedding of G' contains a non-split link with an unused vertex.

□

3 Proof that the operations work

If we let A and B be two cycles that share at least one edge, and whose intersection is connected, then we can define a new cycle, $A + B$ to be the cycle whose edges consist of the disjoint union of edges in A and B. We need the following elementary lemma concerning linking number:

Lemma 3.1. *[4] Let A and B be two cycles that share at least one edge, and whose intersection is connected. Let C be a cycle that is disjoint from both A and B. Then* $lk(A+B,C) = lk(A,C) + lk(B,C)$.

Theorem 3.1. *Any graph obtained from an intrinsically linked graph by the addition of a triangle along three non-adjacent verticies will be intrinsically linked with an extra vertex.*

Proof. Let us begin with an intrinsically linked graph G, embedded in $\mathbb{R}^3$, containing linked cycles α and ζ. Suppose further that the linked cycles use all verticies in the graph, for otherwise we already have a graph with linked cycles and an unused vertex. Now pick any three mutually non-adjacent verticies v_1, v_2, and v_3. By the pigeon hole principle, two of these verticies will lie along the same linked cycle. Without loss of generality, let's say that v_1 and v_2 lie in α. Since the two verticies are now connected, we can break down α into a sum of cycles: $\alpha = \alpha' + \alpha''$. By Lemma 3.1, we can further say that $lk(\alpha,\zeta) = lk(\alpha',\zeta) + lk(\alpha'',\zeta)$. By assumption, $lk(\alpha,\zeta) \neq 0$, and as such the linking of α' with ζ must be non-zero or the linking number of α'' with ζ must be non-zero. Therefore, if α'' is linked there will exist an unused vertex among the vertices of α' or visa versa if α' is the linked cycle.

□

This techniques creates $K_{3,1,1,1,1}$ out of $K_{3,3,1}$. Note that in [1] it is shown that $K_{3,1,1,1,1}$ is minor-minimal with respect to being intrinsically linked with an unused vertex. In addition, $K_{3,1,1,1,1}$ does not contain a disjoint copies of an intrinsically linked graph and a vertex. (If it did, then the intrinsically linked graph would have to have 6 vertices, hence be K_6, but $K_{3,1,1,1,1}$ does not contain K_6 as an induced subgraph.) In [6], it was shown that $\Delta - Y$ exchanges preserve intrinsic linking. By similar reasoning, $\Delta - Y$ exchanges preserve intrinsic linking with an unused vertex. We note here that all graphs obtained from triangle addition on graphs in the Petersen family are either trivial intrinsically linked graphs with an unused vertex, or they can be obtained from $\Delta - Y$ exchanges on $K_{3,1,1,1,1}$. This can be seen as follows. Clearly, triangle addition cannot be applied to K_6. If we examine the graph G_7, which is obtained from a single $\Delta - Y$ exchange on K_6, then the only three mutually non-adjacent verticies lie on a Y. In $K_{4,4} - e$, any three mutually non-adjacent verticies also lie on a Y. One obtains $K_{3,1,1,1,1}$ by triangle addition on $K_{3,3,1}$, and any of the remaining graphs in the Petersen family

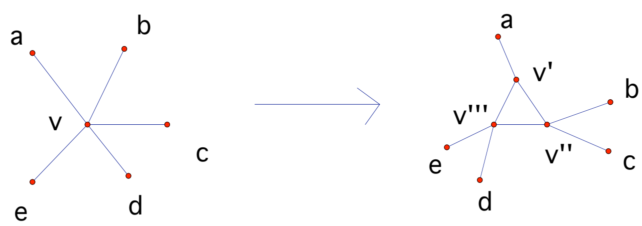

Figure 1: Triangle expansion.

can be obtained from $K_{3,3,1}$ by $\Delta - Y$ exchanges. Thus, any graph obtained from triangle addition on the remaining Petersen graphs, could have been obtained by $\Delta - Y$ exchanges on $K_{3,1,1,1,1}$.

Now we discuss the second move, triangle expansion (see Figure 1, and Figure 3):

Theorem 3.2. *Any graph obtained from an intrinsically linked graph by triangle expansion will be intrinsically linked with an extra vertex.*

Proof. Begin with an intrinsically linked graph, G, and choose a vertex, v, of degree greater than two. We now expand v twice. If after the first expansion one of the resulting two verticies, v or v' are of degree two, then that vertex will not be eligible for initiating the second expansion. Finally, the initial vertex, v, after the two expansions has become a path on three verticies, v, v', and v''. Add an edge from the vertex at one end of the path to the vertex at the other end of the path, resulting in a new graph, G'. The end result is that the vertex v has been replaced with the triangle through v, v' and v''.

Now, once this operation is completed, we embed G'. We can induce an embedding of G from this embedding of G' by shrinking two of the three new edges between v, v', and v'' back to the original vertex, v. We know that G is intrinsically linked independent of the expanded edges from the original vertex, v. Therefore this embedding of G will have a pair of linked cycles, call them A and B. At this point, we can re-expand the vertex v into the triangle containing v, v', and v''. If v was not contained within one of the linked cycles in G, then with the expansion to G' we have three unused verticies. However, if v was contained within one of the linked cycles (without loss of generality, say cycle A), we must now consider the cycle A' in G' that contracts onto A. If one of v, v', or v'' is not in A', we are done. Otherwise, clearly in G', A' is linked with B, that is lk(A',B) $\neq 0$. We denote the three cycle through

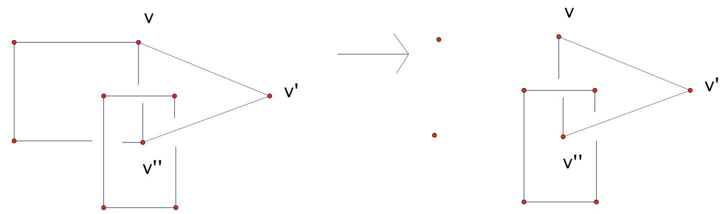

Figure 2: In this case, there are two vertices not used in the link.

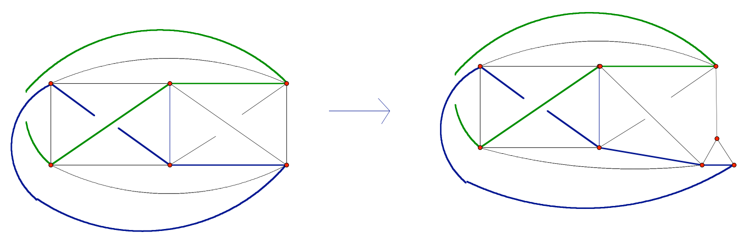

Figure 3: Triangle expansion takes K_6 to a graph that is intrinsically linked with an unused vertex.

v, v', and v'' by C'. Since the intersection of A' and C' is connected, we may form their sum, $A' + C'$. Since $\mathrm{lk}(A'{+}C', B) = \mathrm{lk}(A', B) + \mathrm{lk}(C', B)$ and because we know that A' has an odd linking number with B, either $A' + C'$ or A' must have an odd linking number with B, which will result in at least one extra vertex. This extra vertex will be in A' if $A' + C'$ is linked, or conversely, in $A' + C'$ if A' is the linked cycle (see Figure 2). Thus, we are guaranteed at least one extra vertex in every embedding.

□

As an example of using this operation, we consider the graph obtained from K_6 by a single triangle expansion shown in Figure 3. We will denote this graph G_8. We claim that G_8 does not contain disjoint copies of an intrinsically linked graph and a vertex. Removing a vertex of degree higher than 3 from G_8 results in a graph with fewer than 15 edges, hence a graph that is not intrinsically linked (it follows from [8], that an intrinsically linked graph cannot have fewer than 15 edges). Removing a degree 3 vertex from G_8 results in a graph with exactly 15 edges, but one vertex is of degree 2. Thus the resulting graph is a subdivision of a graph on 14 vertices, hence cannot be intrinsically linked.

Note that since $\Delta - Y$ moves preserves intrinsic linking with an extra vertex, then one could also define another operation "Y expansion," where a vertex is replaced with a Y, and all edges that were incident to the vertex

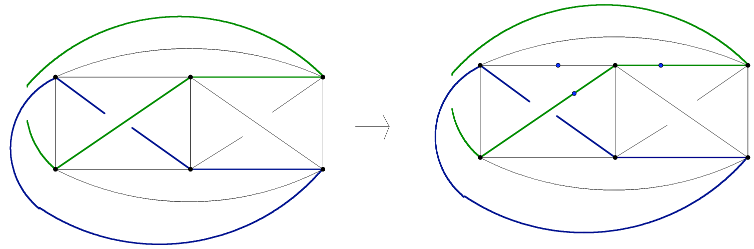

Figure 4: Adding degree 2 vertices to three edges incident to the same vertex transforms K_6 into a minor-minimal intrinsically linked graph with an unused vertex.

are now incident to the Y. Such an operation would create an intrinsically linked graph with an unused vertex out of an intrinsically linked graph.

There are many different graphs that result from performing triangle expansion on the various graphs in the Petersen family. It is a difficult task (one we have not completed) to find all such graphs and then to figure out which of these are minor-minimal with respect to being intrinsically linked with an extra vertex.

Finally, we discuss the third operation, adding degree 2 vertices (see Figure 4):

Theorem 3.3. *Any graph obtained from an intrinsically linked graph by adding degree two verticies along edges that cannot be involved in two disjoint cycles will be intrinsically linked with an extra vertex.*

Proof. If there exists a vertex that is not used up by two disjoint cycles then if those two cycles are linked in an embedding, then there exists an extra vertex. Thus by adding verticies such that no two disjoint cycles can share all the added verticies, at least one vertex will always remain outside of a link.

□

Performing this operation to the seven graphs in the Petersen family creates many minor-minimal graphs that are intrinsically linked with an unused vertex. For example, in K_6, adding degree 2 vertices on three edges incident to the same vertex creates one such graph (see Figure 4).

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