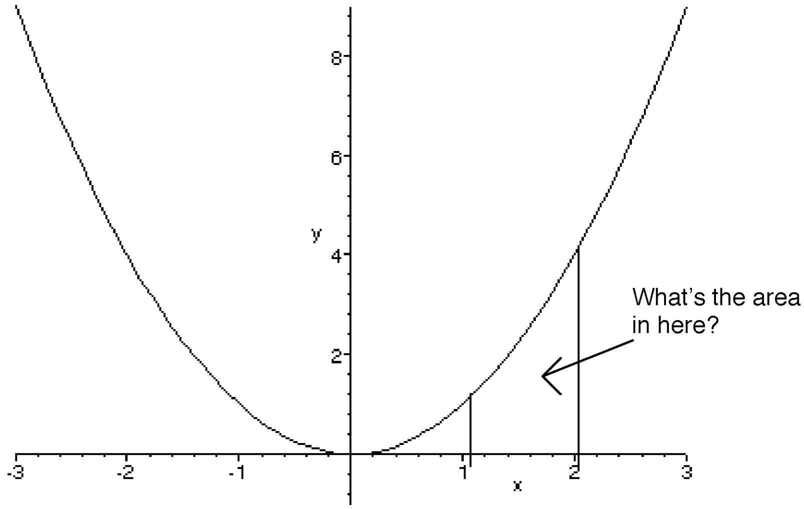

Figure 5.1: How much area is there between the curve $y = x^2$ and the x axis for $1 \leq x \leq 2$?

5.1 Definite Integrals and Reimann Sums

You have been studying differential calculus, which is concerned with derivatives of functions and all of their many applications. Now we start the second part of calculus, the integral calculus. One major motivation for integral calculus comes from computing areas. The incredible fundamental theorem of calculus will relate the integral calculus to the differential calculus.

5.1.1 The Area Problem:

Let's start with an example. Suppose we have the function $y = x^2$, and suppose we want to compute the area between this curve and the $x-$axis between $x = 1$ and $x = 2$ (see figure 5.1).

The key idea is to relate this problem to the problem of computing the area of rectangles-something we can compute easily. Here's how we get to rectangles: cut up the interval from $x = 1$ to $x = 2$ into a bunch of pieces. Over each piece, create a rectangle whose height is $f(x_i)$, where x_i is the right endpoint of the ith interval. If we cut up the interval into n equal length pieces, and denote the length of each subinterval by Δx, we get the total area of all of the approximating rectangles to be $\sum_{i=1}^{n} f(x_i)(\Delta x)$. If we cut up the interval into

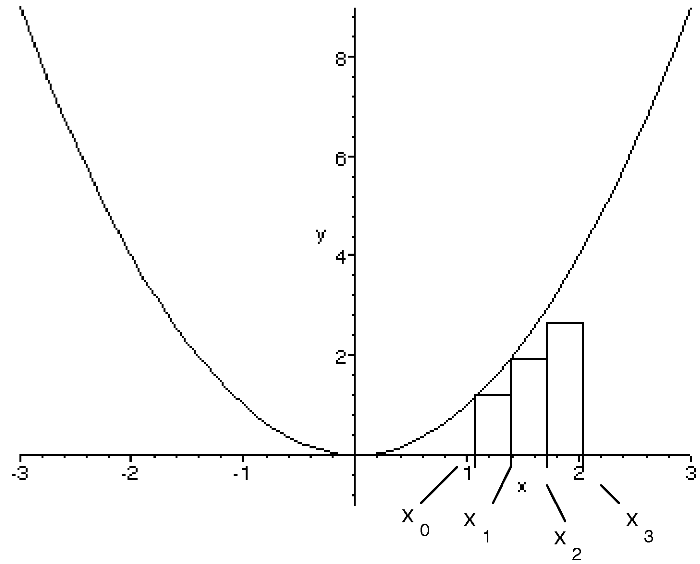

Figure 5.2: Using 3 rectangles to approximate the area between $y = x^2$ and the x axis for $1 \leq x \leq 2$. In this figure, we use $n = 3$ subintervals, and left hand endpoints to determine the rectangle height, $\Delta x = (b - a)/n = 1/3$. We have $x_0 = 1$, $x_1 = 4/3$, $x_2 = 5/3$, and $x_3 = 6/3 = 2$.

smaller pieces,our rectangles give a better approximation of the area under the curve. Under the right conditions on our function, we get the exact area by taking the limit. This limit gets its own name, the definite integral, and is denoted:

$$\int_1^2 x^2 dx = \lim_{n \to \infty} \sum_{i=1}^{n} x_i^2 \Delta x$$

We'll come back to this example shortly. Now the time is right to discuss the general case. In general, if $f(x)$ is a function that is defined over the interval $[a, b]$, under the right conditions, we define $\int_a^b f(x)dx$ as follows. First cut up the interval $[a, b]$ into n subinterval of length $\dfrac{b-a}{n} = \Delta x$.

We'll be particularly concerned with the values of x on the endpoints the created subintervals:

$x_0 = a$
$x_1 = x_0 + \Delta x = a + \Delta x$
$x_2 = x_1 + \Delta x = x_0 + 2\Delta x = a + 2\Delta x$
$x_3 = x_2 + \Delta x = x_0 + 3\Delta x = a + 3\Delta x$
.
.
$x_i = a + i\Delta x.$
.
$x_n = x_{n-1} + \Delta x = x_0 + n\Delta x = x_0 + n(\frac{b-a}{n}) = b$

Now we define the right-hand endpoint Riemann sum for n equal subintervals to be:

$$R(n) = \sum_{i=1}^{n} f(x_i)\Delta x$$

and the left-hand endpoint Riemann sum for n equal subinterval to be:

$$L(n) = \sum_{i=1}^{n} f(x_{i-1})\Delta x$$

If $\lim_{n\to\infty} R(n)$ exists, define $\int_a^b f(x)dx = \lim_{n\to\infty} R(n)$. We call this the *definite integral of f(x) from a to b.* We point out that this limit always exists when the function f is continuous. It is also true that if $\lim_{n\to\infty} R(n)$ exists, then so does $\lim_{n\to\infty} L(n)$. You may see the proof of these facts in a later course.

Before we compute some definite integrals, let's compute some Riemann sums for the function $y = x^2$ over the interval from 1 to 2. Let's use $n = 2$, with right-hand endpoints. We have $a = 1$, $b = 2$, and $\Delta x = 1/2$. We thus have:
$x_0 = 1$
$x_1 = x_0 + \Delta x = 1 + 1/2 = 3/2$
$x_2 = x_1 + \Delta x = 3/2 + 1/2 = 2$

Now we're ready to compute $R(2) = \sum_{n=1}^{2} x_k^2(1/2) = (3/2)^2(1/2)+2^2(1/2) = 9/8 + 2 = 3.125$. How do you think this computation relates to the actual area?

Now, let's use $L(4)$ to approximate the area of the same region. Again, we have $a = 1$ and $b = 2$, but this time $\Delta x = 1/4$. We thus have:

$$\begin{aligned}
x_0 &= 1 \\
x_1 &= x_0 + \Delta x = 1 + 1/4 = 5/4 \\
x_2 &= x_1 + \Delta x = 5/4 + 1/4 = 3/2 \\
x_3 &= x_2 + \Delta x = 3/2 + 1/4 = 7/4 \\
x_4 &= x_3 + \Delta x = 7/4 + 1/4 = 2
\end{aligned}$$

From this, we get $L(4) = \sum_{n=1}^{4} x_{k-1}^2(1/4) = 1^2(1/4)+(5/4)^2(1/4)+(3/2)^2(1/4)+(7/4)^2(1/4) = 1/4 + 25/64 + 9/16 + 49/64 = 1.96875$. This is smaller than our previous estimate, is that a coincidence?

OK, let's stop approximating, and let's try to compute the exact area. This will take a little more work, but we can do it! In fact, we'll need two of the following formulas:

$$\boxed{\begin{aligned}
\sum_{i=1}^{n} i &= \frac{n(n+1)}{2} \\
\sum_{i=1}^{n} i^2 &= \frac{n(n+1)(2n_+1)}{6} \\
\sum_{i=1}^{n} i^3 &= \left(\frac{n(n+1)}{2}\right)^2
\end{aligned}}$$

Now let's take a look at what we have to compute:

$$\int_1^2 x^2 dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

This time, we leave Δx in the mysterious form $(b-a)/n$, which can be simplified to $1/n$, since $b-a=1$. One difficult task is to figure out a formula for x_i. It is still true that $x_0 = a = 1$. We get $x_1 = x_0 + \Delta x = 1 + 1/n$. Similarly, $x_2 = x_1 + \Delta x = 1 + 1/n + 1/n = 1 + 2/n$. In general, we have $x_i = 1 + i/n$. Now we need to expand and simplify $R(n)$, and later we'll take its limit:

$$R(n) = \sum_{i=1}^{n}(1+i/n)^2(1/n). \qquad \text{FOIL this out to get:}$$

$$= \sum_{i=1}^{n}(1+2i/n+i^2/n^2)(1/n). \qquad \text{Simplify to get:}$$

$$= \sum_{i=1}^{n}(1/n+2i/n^2+i^2/n^3). \qquad \text{Split this up:}$$

$$= \sum_{i=1}^{n}(1/n)+\sum_{i=1}^{n}2i/n^2+\sum_{i=1}^{n}i^2/n^3). \qquad \text{Factor out terms not involving } i\text{:}$$

$$= (1/n)\sum_{i=1}^{n}1+(2/n^2)\sum_{i=1}^{n}i+(1/n^3)\sum_{i=1}^{n}i^2. \qquad \text{Now use the fancy formulas from above:}$$

$$= (n/n)+\left(\frac{2}{n^2}\right)\left(\frac{n(n+1)}{2}\right)+\left(\frac{n(n+1)(2n+1)}{6n^2}\right).$$

$$= 1+\frac{2n^2+2}{2n^2}+\frac{(n^2+n)(2n+1)}{6n^3}$$

$$= 1+\frac{2n^2+2}{2n^2}+\frac{2n^3+3n^2+n}{6n^3}$$

$$= 1+1+1/n^2+1/3+1/(2n)+1/(6n^2)$$

So we get $R(n) = 1+1+1/n^2+1/3+1/(2n)+1/(6n^2)$. Now let n approach infinity. We get $1+1+1/3 = 7/3$. This was a bit less than $R(2)$ and a bit more than $L(4)$. Does that surprise you?

5.1.2 The Velocity Problem:

The velocity problem is another main motivation for the integral calculus. Suppose you drive down the highway at 60 miles per hour, on cruise control, for a half an hour. How far did you travel during that half an hour? 30 miles. For a body traveling at a *constant* velocity, v, over a given time, t, the total distance traveled is the product of velocity and time: $d = vt$. It is when velocity is not constant that calculus comes in handy. Suppose the velocity of a car was recorded at regular intervals, and the data put into the following table:

time(hour)	0	.1	.2
velocity(miles/hour)	10	30	40

Now, the car did not travel at a constant speed, but we will assume that the speed of the car is constant over each interval. So, over the first interval (.1 hours), we'll assume that the car travelled 10 miles per hour, the speed at the beginning of the interval. This is a bit arbitrary, but it will turn out to be analogous to using left hand endpoints to figure out Reimann sums. So, over the first .1 hour, the car traveled about $(.1)(10) = 1$ mile. Over the second (last) interval, the car traveled about $(.1)(30) = 3$ miles. The total distance traveled is approximately $1 + 3 = 4$ miles. If we want a more accurate measure of how far the car traveled, we could record the velocity at shorter intervals. To get more precise, limits will be required.

Hopefully, given our previous discussion, the following definition won't surprise you. Given a velocity function $v = f(t)$, with $f(t) \geq 0$, and $f(t)$ continuous, define **the total distance traveled** from $t = a$ to $t = b$ to be $\lim_{n\to\infty} \sum_{i=1}^{n} f(t_i)\Delta t = \lim_{n\to\infty} \sum_{i=0}^{n-1} f(t_i)\Delta t$. Where $\Delta t = \dfrac{b-a}{n}$, and $t_i = a + i\Delta t$. When $f(t)$ is always greater than or equal to 0, this is exactly equal to the area between the graph of $y = f(t)$ and the t axis for $a \leq t \leq b$.