Chapter 2

Limits and Rates of Change

2.1 The Tangent and Velocity Problems

Now we are starting to get to the heart of calculus. Two fundamental problems are finding tangents to curves and finding the velocity of an object.

2.1.1 Finding tangents to curves

For a circle, a tangent line is a line that intersects the circle in just one point.

For a more general curve, this does not characterize tangent lines (see Stewart, p. 67). A good intuitive way to think about tangents is that if you imagine yourself driving along the curve, and you turn on your headlights, your headlights would point in the same direction as the tangent line.

In some sense, a tangent line to a curve at a point should have the same slope as the curve. The problem is, what do we mean when we talk about the slope of a curve? For a line, the slope is the same no matter what points we pick to determine Δy and Δx. This is not true for a curve that is not a line. For such a curve (provided it is "smooth"), we can get around this by looking at secant lines through the point x, and a nearby point, say $x + \Delta x$. If we let Δx go to zero, the slopes of the secant lines should approach the slope of the tangent line. Let's look at an example:

Example: Estimate the slope of the tangent line to the curve $y = 2 - x^2$ at the point $(1, 1)$.

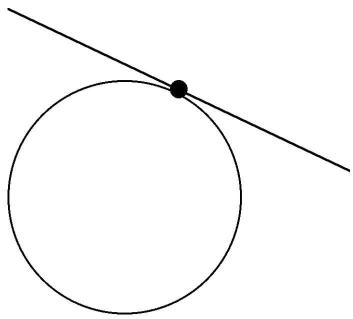

Figure 2.1: A tangent to a circle.

Δx	$x + \Delta x$	$f(x + \Delta x)$	$\dfrac{f(x + \Delta x) - f(x)}{\Delta x}$
.1	1.1	.79	−2.1
.01	1.01	.9799	−2.01
.001	1.001	.997999	−2.001
−.1	.9	1.19	−1.9
−.01	.99	1.0199	−1.99
−.001	.999	1.001999	−1.999

It seems that the slopes of the secants are approaching -2, as we let our second point approach our first. The first three entries on our table have $x + \Delta x$ to the right of x, and the last three have $x + \Delta x$ to the left. Later, we shall have the necessary tools at hand to prove that the slope of the tangent line to $y = 2 - x^2$ at $x = 1$ is indeed -2.

2.1.2 Instantaneous Speed

Suppose you go to Canton in 10 minutes (=1/6 of an hour). Canton is 10 miles away, so your average speed in miles per hour is: (10 miles)/(1/6 hour) = 60 miles/hr. In general, average speed is (change in distance)/(change in time).

Example 3 in your text (p. 70) is a good one to look at. (Even though the authors bias for Canada is blatantly obvious here.) In that example, a ball is dropped from a tower. How fast is the ball traveling after exactly 5 seconds? What about after 6 seconds?

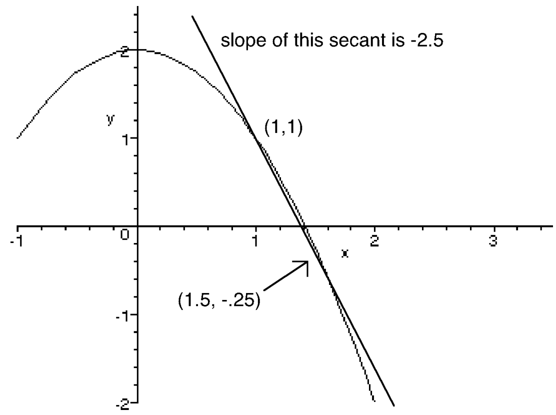

Figure 2.2: The secant line through $(1, 1)$ and $(1.5, -.25)$.

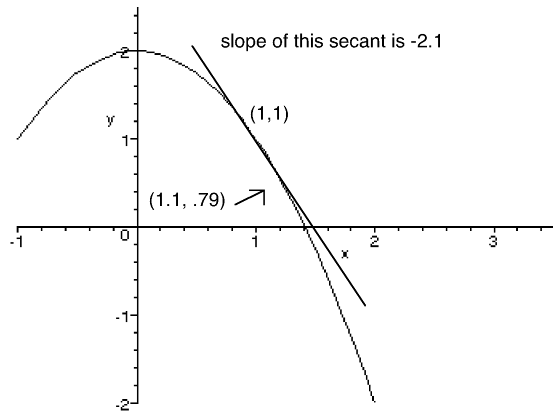

Figure 2.3: The secant line through $(1, 1)$ and $(1.1, .79)$.

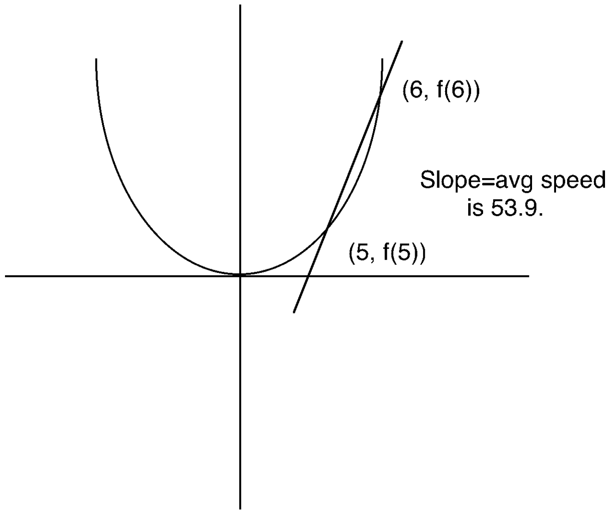

Figure 2.4: The secant line whose slope represents average speed from $t = 5$ to $t = 6$.

By Galileo's law, the ball will have fallen a distance $s(t) = 4.9t^2$ meters after t seconds. We can approximate the velocity traveled at 5 seconds by computing the average velocity over intervals around 5. For example, the distance the ball has fallen at $t = 5$ is $4.9(5^2) = 122.5$ meters. After 6 seconds, the ball has fallen $4.9(6^2) = 176.4$ meters. So, between the fifth and sixth seconds, the ball has fallen $176.4 - 122.5 = 53.9$ meters. The average speed over that second is 53.9 meters/second. If we shorten up the interval to start at $t = 5$ and end at $t = 5.1$, we can compute the average speed to be 49.49 meters/second (see Stewart, p. 70). Similarly, we can compute the average speed between $t = 6$ and $t = 6.1$. We get:

$$\frac{s(6.1) - s(6)}{0.1}$$

$$= \frac{4.9(6.1)^2 - 4.9(6)^2}{0.1} = 59.29.$$

Are you surprised that the speed at 6 seconds seems to be faster than the speed at 5 seconds? Have you ever bungee jumped or jumped off a cliff into a river? The farther you fall, the faster you travel.

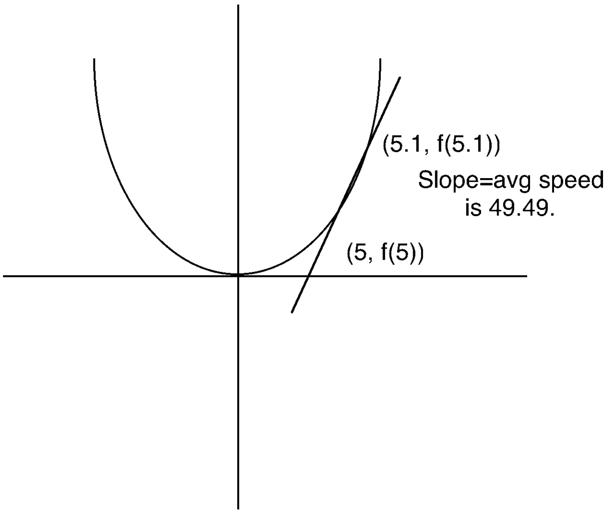

Figure 2.5: The secant line whose slope represents average speed from $t = 5$ to $t = 5.1$.

Length of interval	Avg speed, starting at $t = 5$	Avg. speed starting at $t = 6$
1	53.9	63.7
.1	49.49	59.29
.05	49.245	59.045
.001	49.0049	58.8049

It seems that the instantaneous speed at $t = 5$ is approaching 49, and the instantaneous speed at $t = 6$ is approaching perhaps 58.8. To be sure of a precise answer, we need to study some more of the calculus coming up! In particular, we need to explore the topic of limits.

As the author states on pp. 69-70, the tangent problem and the instantaneous velocity problem are really the same thing. The average speed quotients in the second example can also be viewed as representing slopes of secant lines. As we let the change in time approach zero, we are making the second point on the secant line approach the first.

2.2 The Limit of a Function

We saw from trying to find instantaneous speed that as we shorten our interval length, the average speed seems to be approaching a fixed number. We could also consider the function $f(x) = \dfrac{x^4 - 1}{x - 1}$. As you saw in lab, the values

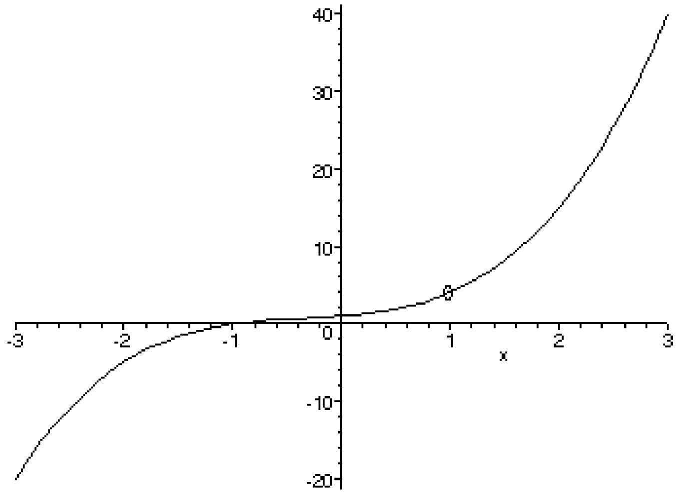

Figure 2.6: As x approaches 1, $g(x)$ approaches 4, even though $g(1) = 1000$.

of $f(x)$ seem to be approaching 4 as values of x approach 1, even though 1 is not in the domain of f. A quick table can be made:

x	$f(x)$	x	$f(x)$
.99	3.940399	1.01	4.060401
.999	3.99403999	1.001	4.006004001

Yup, the values of $f(x)$ seem to approach 4, and in fact we can write $\lim_{x\to 1} \frac{x^4 - 1}{x - 1} = 4$.

Let's write out an informal definition of limit (see Stewart, p. 71), a more formal definition will come later:

Let $f(x)$ be defined on an open interval about a, except possibly at a itself. We write

$$\lim_{x\to a} f(x) = L$$

and say "the **limit** of $f(x)$ as x approaches a is L" if we can guarantee the values of $f(x)$ arbitrarily close to L (as close to L as we like) by choosing x to be sufficiently close to a, but not equal to a.

Important note: the value of $\lim_{x\to a} f(x)$ does not depend at all on the value of $f(x)$ at $x = a$. In fact, $f(x)$ does not even have to be defined for $x = a$. For example, consider the following function:

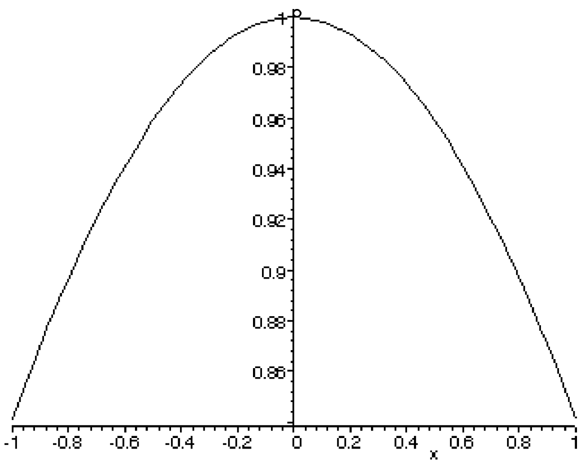

Figure 2.7: One can make $f(x) = \dfrac{\sin x}{x}$ arbitrarily close to 1 provided x is sufficiently close to 0.

$$g(x) = \begin{cases} (x^4 - 1)/(x - 1) & \text{if } x \neq 1 \\ 1000 & \text{if } x = 1 \end{cases}$$

We could make a table for this, but it would have the same entries as the table above, and we have $\lim_{x \to 1} g(x) = 4$. You see, the value of $g(1)$ is quite different from the limit of $g(x)$ as x approaches 1.

You should consult your text for some nice examples of what can go wrong when using the calculator to estimate limits (see example 2, p. 72 in particular).

Another important example is $\lim_{x \to 0} \dfrac{\sin x}{x}$. You can graph this on your calculator (in radian mode) and use the trace button to guess what the limit is. Stewart gives a nice table on p. 73. The limit is in fact 1, which your teacher may prove later in the semester (see figure 2.7).

2.2.1 When limits fail to exist.

Is it possible for a limit not to exist? Yes. Consider the following function:

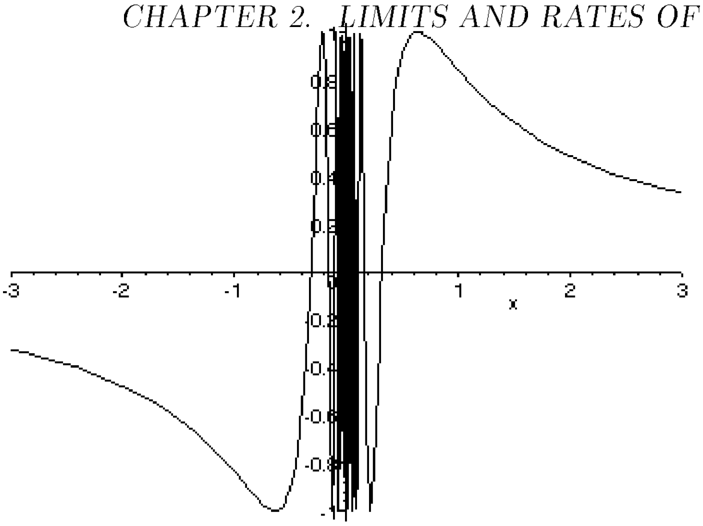

Figure 2.8: This function has no limit as x approaches 0.

$$f(x)=\begin{cases} \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

This function has no limit as $x \to 0$. The function's values oscillate too much near 0 (see figure 2.8).

Here is another, simpler, example of a function with a limit failing to exist at a certain point.

$$H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$$

Let's examine $\lim_{x \to 0} H(t)$. We can even make a table with values of t close to 0:

t	$H(t)$	t	$H(t)$
.1	1	$-.1$	0
.01	1	$-.01$	0
.001	1	$-.001$	0

Look at the graph of this function in figure 2.9. It seems that the values of $H(t)$ aren't approaching a single value as t gets closer and closer to 0. This brings us to the next topic.

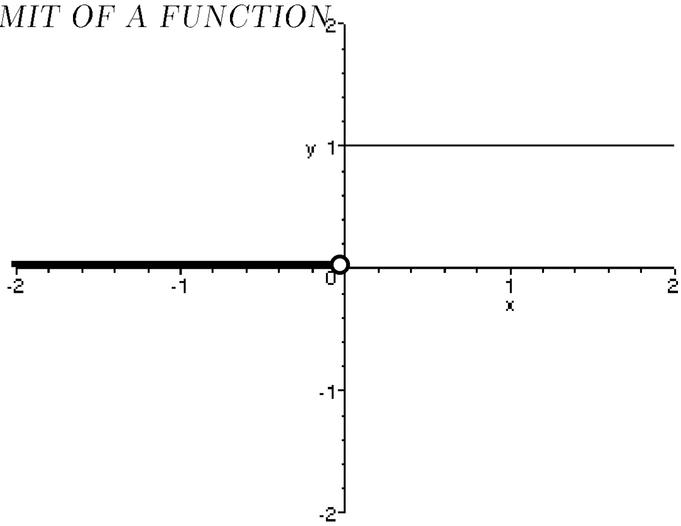

Figure 2.9: This function has no limit as t approaches 0, but it does have a left hand limit and a right hand limit at $t = 0$.

2.2.2 One-Sided Limits

It is true that if we restrict our values of t to those to the left of 0, then $H(t)$ values are 0. We can say that $H(t)$ approaches 0 as t approaches 0 from the left, and we write: $\lim_{x \to 0^-} H(t) = 0$. It is also true that $\lim\limits_{x \to 0^+} H(t) = 1$. Here's an informal definition of left and right hand limits:

Let $f(x)$ be defined on an open interval (c, a), where $c < a$. We write

$$\lim_{x \to a^-} f(x) = L$$

and say **the limit of** $f(x)$ **as** x **approaches** a **from the left** is equal to L, or that L is the **left hand limit** of $f(x)$ as x approaches a if we can guarantee the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a and less than a.

This definition is just like that for the ordinary limit, only the x values must be less than a.

The informal right hand limit definition is similar:

Let $f(x)$ be defined in an open interval (a, b), where $a < b$. We write

$$\lim_{x \to a^+} f(x) = L$$

and say **the limit of** $f(x)$ **as** x **approaches** a **from the right** is equal to L, or that L is the **right hand limit** of $f(x)$ as x approaches a if we can guarantee the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a and greater than a.

You should convince yourself that $\lim_{x \to 0} \sqrt{x}$ does not exist (because points to the left of 0 are not in the domain), but that $\lim_{x \to 0^+} \sqrt{x} = 0$.

By looking at the definitions, you should be able to convince yourself that **the ordinary limit of a function at a point exists if and only if the left and right hand limits exist at that point, and they agree in value**. We state this more formally as:

$\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^+} = L = \lim_{x \to a^-}$.

From this, we may conclude that $\lim_{x \to 0} \frac{|x|}{x}$ does not exist, since its right hand limit does not equal its left hand limit.

Example 7 on p. 76 is a good one to look at right now.

2.2.3 Infinite Limits

Consider values of the function $f(x) = \frac{1}{(x-2)^2}$ for values of x near 2. We can make $f(x)$ as large as we want, provided we pick x close enough to 2. Want $f(x)$ larger than 9999? You could pick $x = 2.01$. Want $f(x)$ larger than 1 million? You could pick $x = 2.0001$. (Try it, $f(2.0001) = 100000000$.) We could go on like this forever, but let's stop now. In fact, we can say that $\lim_{x \to 2} \frac{1}{(x-2)^2} = +\infty$. This doesn't mean that the limit of the function exists at $x = 2$; it doesn't. It means that our function satisfies the following definition:

Let $f(x)$ be a function that is defined for values of x in an interval containing a, except possibly not at a. We write $\lim_{x \to a} f(x) = +\infty$ and say **the limit of $f(x)$ as x approaches a is positive infinity**, if we can ensure the values of $f(x)$ are arbitrarily large by choosing x sufficiently close to a, but not equal to a.

A related notion is that of a limit of negative infinity. We write $\lim_{x \to a} f(x) = -\infty$ and say **the limit of $f(x)$ as x approaches a is negative infinity**, if we can guarantee that the values of $f(x)$ are arbitrarily large and negative by choosing x sufficiently close to a, but not equal to a.

One can define in a similar way left and right hand limits that go to infinity.

2.2.4 Vertical Asymptotes

If the distance between the graph of a function and some fixed line approaches zero as the graph moves increasingly far from the origin, we say that the line is an **asymptote** of the graph.

There is another (but equivalent) definition of vertical asymptote, here it is:

A line $x = a$ is a **vertical asymptote** of the graph of the function $y = f(x)$ if either

$$\lim_{x \to a^+} f(x) = \pm\infty$$

or

$$\lim_{x \to a^-} f(x) = \pm\infty$$

Of course $x = 2$ is a vertical asymptote for the function $\frac{1}{(x-2)^2}$. Note that as x gets close to 2, the denominator of this function approaches 0, and dividing by a number close to zero yields a large (possibly large and negative or positive) number. One can use similar reasoning to justify that $\lim_{x \to 4^-} \frac{1}{x-4} = -\infty$ (see table below for intuitive justification), and thus that $x = 4$ is an asymptote for this function.

x	$f(x)$
3.9	-10
3.99	-100
3.999	-1000
3.9999	-10000

The functions $f(x) = \tan x$ and $g(x) = \sec x$ have vertical asymptotes at odd-integer multiples of $\pi/2$, where $\cos x = 0$ (see Stewart, p. 79 for more details).

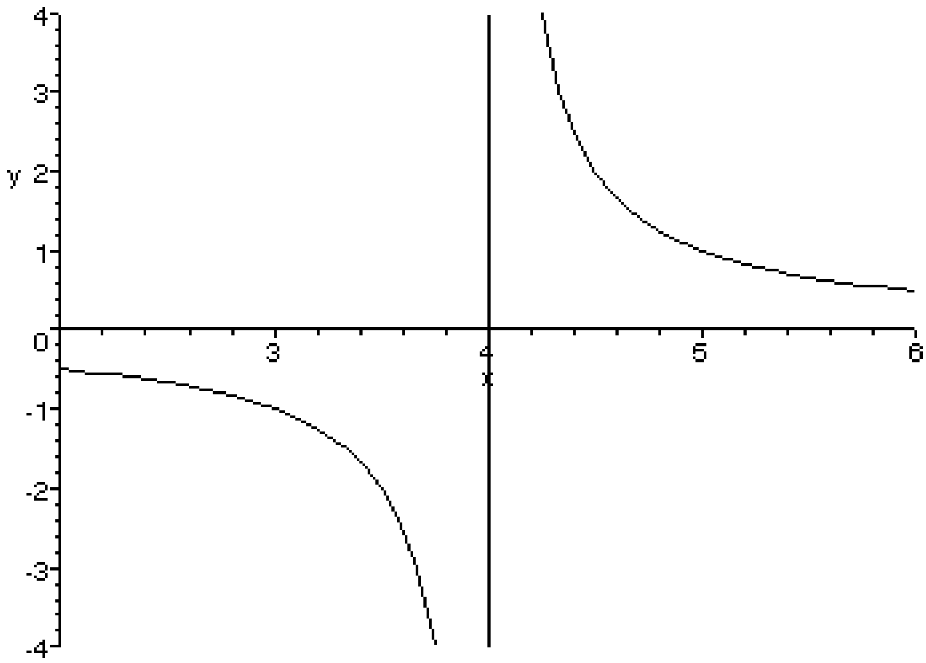

Figure 2.10: $f(x) = 1/(x-4)$ has an asymptote at $x = 4$.