Chapter 1

Functions

1.1 Introduction

We start with a definition:

A **function** from a set D to a set R is a rule that assigns a unique element $f(x)$ in R to each element x in D.

The set D is called the **domain**. It consists of all possible inputs of f. The set R is called the **range**, and it consists of all possible values of $f(x)$ as x varies throughout D. A nice way to think of f is as a machine that takes in x and spits out $f(x)$.

Are the following functions?

1. Let D = set of all living humans. For x in D, define $f(x)$ to be x's biological mother. Is f a function?

   Yes, since every human has a unique biological mother.

2. Let D = set of all mothers, and for x in D, define $f(x) = x$'s child. Is this f a function?

   No, since a mother may have more than one child, this is not a function.

3. Let D = set of all real numbers. For x in D, define $f(x) = 2x + 6$. Is this f a function?

   Yes, since every x is assigned the unique value of $2x + 6$.

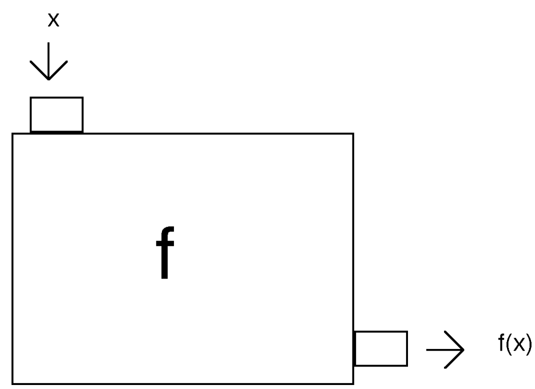

Figure 1.1: The function takes in x and spits out $f(x)$.

4. Let D = the set of all non-negative real numbers. For x in D, define $f(x) = \pm\sqrt{x}$. Is this a function?

   For each positive x, there are two choices for $f(x)$, thus f is not a function. A function has to assign a unique value to x; it cannot assign two values to x.

   If we consider the graph consisting of all ordered pairs that satisfy the equation $y = \pm\sqrt{x}$, we see that above any positive value of x, there are two different values of y. The two points are $(x, +\sqrt{x})$, and $(x, -\sqrt{x})$. Since there are two choices for y values, again we see that this is not the graph of a function. Some vertical lines intersect the graph in two points. We say such a graph fails the vertical line test.

   The Vertical Line Test A curve in the plane is the graph of a function of x if and only if no vertical line intersects the graph more than once.

5. Let D = the set of all real numbers. Define $f(x) = 6$. Is this a function?

   Every value of x is assigned the unique value of 6, so yes this is indeed a function.

6. Let D = set of all humans who own at least one car. Define $f(x) = x$'s car. Is this a function?

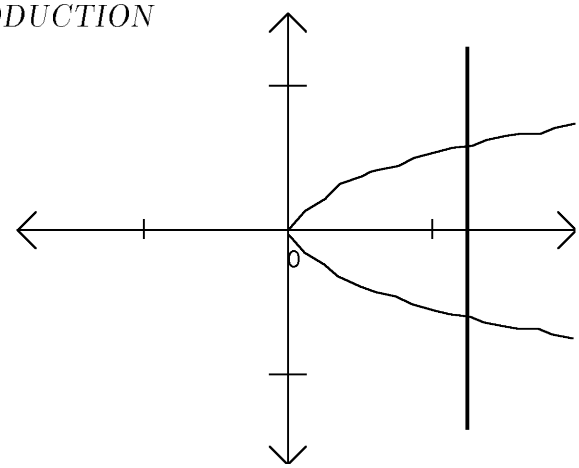

Figure 1.2: There is a vertical line that intersects the graph of $y = \pm\sqrt{x}$ in more than one point.

> No, because there are people who own more than one car. For such a person, it wouldn't be clear what value f should assign. The Lexus? The BMW? Functions never force us to make such choices; functions assign a unique value.

Virtually all of the functions we will deal with from now on will involve numbers. We will, however, look at such functions from four perspectives: *algebraically* (by a formula), *verbally* (in words), in a *table*, and *graphically*.

1. **By formula and graphically:** As we saw above, $f(x) = 6$ and $f(x) = 2x + 6$ are good examples. We can also consider the function $f(x) = +\sqrt{x}$. What should we pick for domain? Since the square root is not defined for negative real numbers (unless you know about complex numbers, but let's not get into that here), let's leave them out. So the domain is $\{x|x \geq 0\}$. What is the range? Can you tell from looking at the graph (see figure 1.3) Remember, the **graph** of a function f consists of all ordered pairs (x, y) in the coordinate plane such that $y = f(x)$, and x is in the domain of f (see p. 12 of Stewart).

   Let's consider the function $f(x) = 1/x$. What would be an appropriate domain for this? Well, we never, ever want to divide by 0, so if we leave out 0 we'll be ok. Thus the domain is $\{x : x \neq 0\}$. What is the range? Again, you should look at the graph and figure out the values

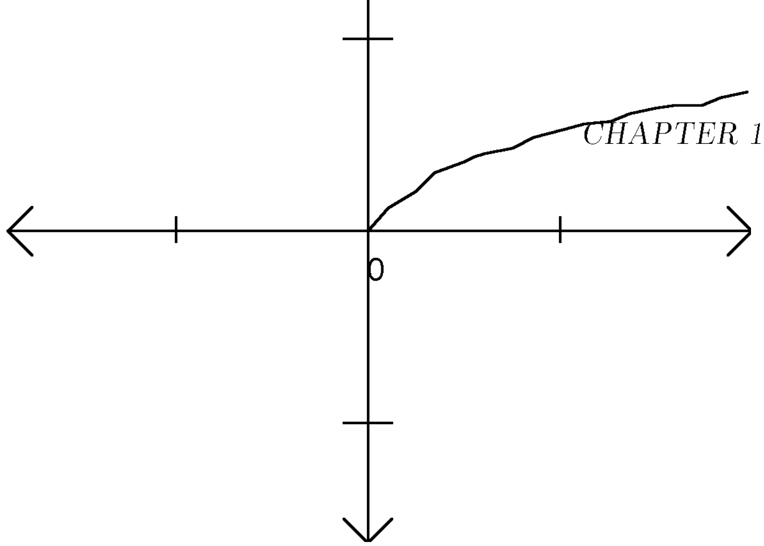

Figure 1.3: A (bad) sketch of the graph of $y = \sqrt{x}$. What $y-$values does this graph take on?

y takes on.

Rule of thumb for finding domain of a function given by a formula: Avoid values of x that give you a zero in the denominator, and avoid values of x that give you a negative inside of a square root.

2. **Verbally and by table** Here's an example of a function given by words: the number of student in Carson 205 as a function of t, where t consists of selected times of the day on September 4, 1999. We could also use a table to describe such a function:

time= t	number of students= $f(t)$
8:15	22
9:15	28
10:15	31
11:15	37
12:15	40
1:15	32
2:15	34
3:15	12
4:15	18
5:15	12
6:15	5
7:15	0

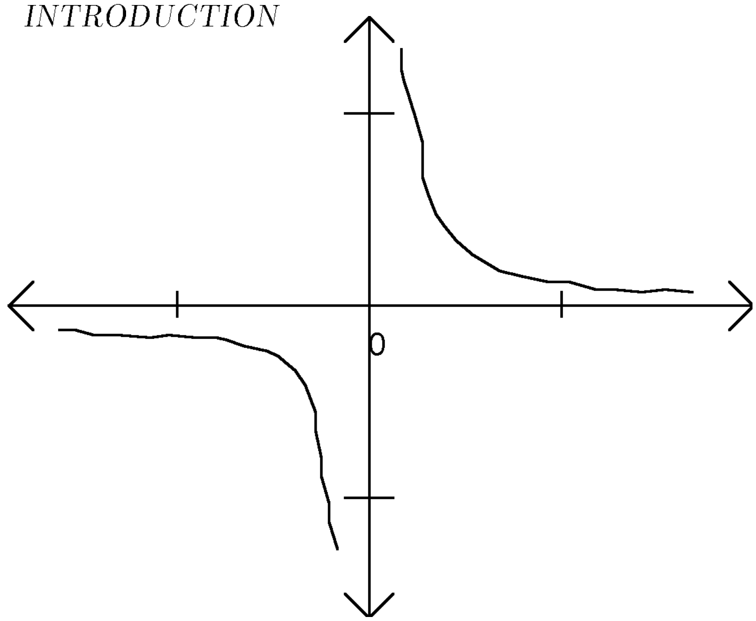

Figure 1.4: A (bad) sketch of the graph of $y = 1/x$.

3. **Verbally and graphically** The height of a given blade of grass in the quad at SUNY Potsdam as a function of time. What should this graph look like (see figure 1.5)?

1.1.1 Piecewise Defined Functions

You've seen an example of a piecewise defined function before, namely the absolute value function. Here's another example:

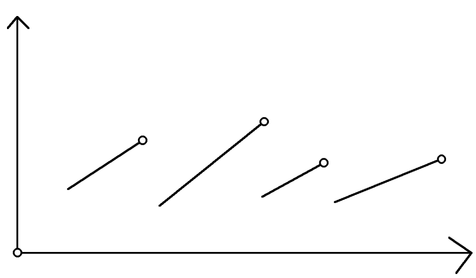

Figure 1.5: The height of a blade of grass over a few month's time.

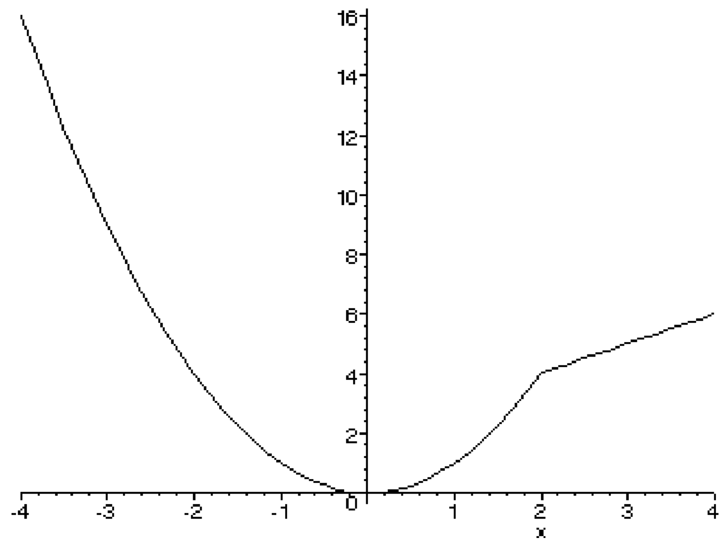

Figure 1.6: A piecewise defined function.

$$f(x) = \begin{cases} x+2 & \text{if } x \geq 2 \\ x^2 & \text{if } x < 2 \end{cases}$$

This function is defined for all real values of x, but the rule you use to determine $f(x)$ depends on x's position on the real line.

Let's try another example of a piecewise defined function:

$$f(x) = \begin{cases} x+4 & \text{if } x \geq 2 \\ x^2 & \text{if } x < 2 \end{cases}$$

1.1.2 Symmetries

Graphs can have various symmetries. Here we discuss two nice types of symmetries. First, an **even** function is one that is symmetric about the $y-$axis. In other words, you could fold the graph along the $y-$axis, and the graph would overlap itself. The function $y = x^2$ has this type of symmetry (see figure 1.7).

A graph that is symmetric about the origin is said to have **odd** symmetry. This means you could rotate the graph about the origin 180 degrees, well π radians, and the graph overlaps itself. The function $y = x^3$ has this property (see figure 1.7).

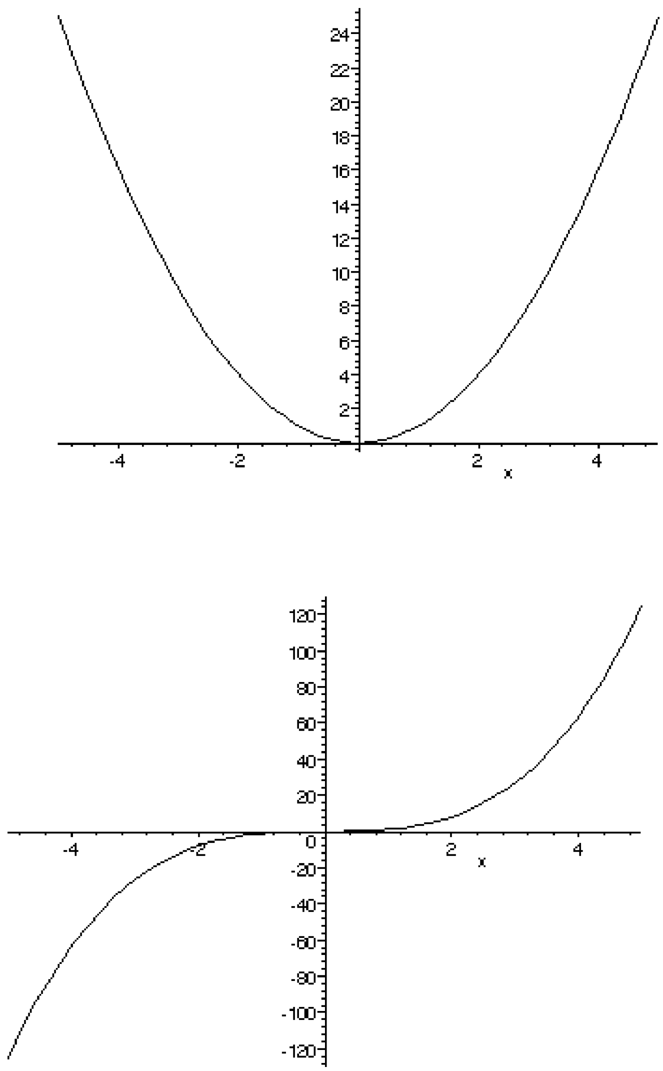

Figure 1.7: $y = x^2$ is even , while $y = x^3$ is odd .

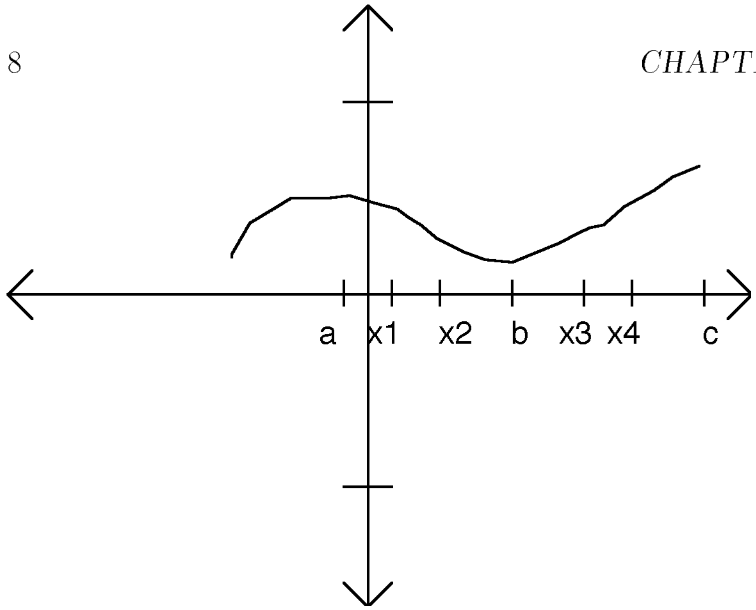

Figure 1.8: The function is decreasing on (a, b) and increasing on (b, c).

There is a more formal definition of even and odd. We give it here: a function f is **even** is $f(-x) = f(x)$ for all x in the domain of f. The function f is **odd** if $f(-x) = -f(x)$ for all x in the domain of f.

For fun, let's check that $f(x) = x^2$ is even. Start with $f(-x)$, we want to somehow modify this to be $f(x)$. Well, $f(-x) = (-x)^2$, but the square of a negative is always positive, so $(-x)^2 = x^2 = f(x)$. So we do have that $f(-x) = f(x)$.

1.1.3 Increasing and Decreasing Functions

This topic will become really important after we've studied derivatives. It is good to start thinking about it now.

On an interval, a function is said to increasing if as you increase x values, the corresponding $f(x)$ values also increase. A function is said to be decreasing if as you increase x values, the corresponding $f(x)$ values decrease. More formally, we have the following:

A function f is called **increasing** on an interval I if: for *any* x_1 and x_2 in I with $x_1 < x_2$ you have $f(x_1) < f(x_2)$ It is called **decreasing** on I if: for *any* x_1 and x_2 in I with $x_1 < x_2$ then you have $f(x_1) > f(x_2)$

1.2 Library of Functions

Mathematicians find functions interesting in themselves, but perhaps you need some more justification as to why functions are useful. Applied mathematicians, scientists and social scientists use functions to model real world phenomena. For example, social scientists use population models to predict world population growth. That is, they have computed functions, say $P(t)$, that give the world population given an input of t. So for such a function, if $P(2001) = 50$ billion, then either the function isn't very realistic, or our population is going to grow by about 10 times in the next couple of years, and policy makers should think about this problem. In this way, models are used to guide us as we try to make the world a better place. If a model "stinks" (its predictions prove to be off) then we need to make modifications and improve the model. In order to come up with good models for a variety of phenomena, it pays off to have a lot of functions at our disposal.

1. **Linear Functions** are functions of the form $f(x) = mx + b$. Graphically, they are lines. The historic level of carbon dioxide in the atmosphere can be modelled by a linear function (see p. 27 of Stewart). It's kind of scary how it's going up, isn't it? See how functions can be disturbing. They can also provide good evidence for pollution control.

2. **Polynomials**. A polynomial function $p(x)$ is of the form $p(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2 + a_1$. Where n is an integer greater than or equal to 0, and $a_0, a_1, \ldots, a_n$ are all real constants. The greatest value of n such that $a_n \neq 0$ is called the **degree** of the polynomial (you can think of it as the highest exponent in the expression). A linear function is a polynomial of degree 1 or 0.

   The height of a falling object can be modelled very nicely as a second degree function of time. See p. 30.

3. **Power Functions**. There are three popular types. The first is a function of the form $f(x) = x^n$, where n is a natural number. What happens to the graphs of $y = x^n$ as n increases?

   The second popular example of power functions are those of the form $f(x) = x^{\frac{1}{n}} = \sqrt[n]{x}$. The last example are those of the form $f(x) = \dfrac{c}{x} =$

cx^{-1}. The relationship between the pressure and volume of a gas can be modelled using this type of function, so physicists like this form.

4. **Rational Functions** are the quotients of two polynomials. $f(x) = \frac{p(x)}{q(x)}$. There domain is the set of all x such that $q(x) \neq 0$. Unlike polynomials, they have asymptotes.

5. **Agebraic Functions** can be constructed using algebraic operations (addition, subtraction, multiplication, division, and taking roots) starting with polynomials. Examples include $f(x) = \sqrt{x} + 2x + 1$, $g(x) = x^2 + 7$, $h(x) = \frac{1 + x + 7\sqrt[3]{x^{4+7}}}{2\sqrt{x} + 12x + 2}$. The "doomsday" model for world population is a rational function that seems to fit the historical data very well, and has a vertical asymptote around $t = 2033$. This is another disturbing function.

6. **Trig Functions**. We've discussed these already. They are extremely useful for modelling periodic phenomena.

7. **Exponential Functions** are functions of the form $f(x) = a^x$ where a is a positive constant. We will discuss these in more detail later. They are useful for modelling growth of money in a savings account, and they are useful for modelling radioactive decay (carbon dating).

8. **Logarithmic Functions** are given by $f(x) = \log_a(x)$, where $y = \log_a(x)$ if and only if $a^y = x$. Log functions increase slowly when $x > 1$ (see p. 35). Scientists use the log function a lot. Computer scientists use the log function frequently to analyze the efficiency of an algorith. (An algorithm that works in logarithmic time is very efficient.)

9. **Transcendental Functions** are not algebraic. They include the trigonometric, exponential, and logarithmic functions, in addition to other unnamed functions.

1.3 New Functions From Old Functions

1.3.1 Sums, Differences, Products, and Quotients

We can add two functions together to get a new function. We do this by the rule:

$$(f+g)(x) = f(x) + g(x).$$

For example, suppose that $f(x) = \sqrt{x}$ and $g(x) = x^2$. Let's figure out what $(f+g)(4)$ is. Well, it's $f(4) + g(4) = \sqrt{4} + 4^2 = 2 + 16 = 18$. Not so bad. Of course, the domain of the new function $f+g$ is the intersection of the domain of f with the domain of g, since any x that you want to apply $f+g$ to has to go through both f and g. In our example, the domain of $\sqrt{x} + x^2$ is all non-negative real numbers, since you do not get a real number when you take the square root of a negative number. Subtraction and multiplication work the same way:

$$(f-g)(x) = f(x) - g(x)$$

and

$$(fg)(x) = f(x)g(x)$$

Again, the domains of $f-g$ and fg consist of the intersection of the domain of f with the domain of g. Finally, we can define the function $(f/g)(x) = \dfrac{f(x)}{g(x)}$. The domain of (f/g) is the intersection of the domain of f with the domain of g, with the added condition that x is not in the domain of (f/g) if $g(x) = 0$ (NEVER divide by 0). Consider $f(x) = x^2$, and $g(x) = x$, making the function $(f/g)(x) = \dfrac{x^2}{x} = x$. In this case, we still forbid 0 from the domain of (f/g). Since $g(0) = 0$, we cannot allow it. The graph of $y = (f/g)(x)$ is the same as the graph of $y = x$, except there is a hole at the point $(0, 0)$.

1.3.2 Composition of Functions

This is another way to combine functions. Let's define it first: the composition of the functions f and g is given by

$$(f \circ g)(x) = f(g(x))$$

So, we take x, first apply g, then apply f.

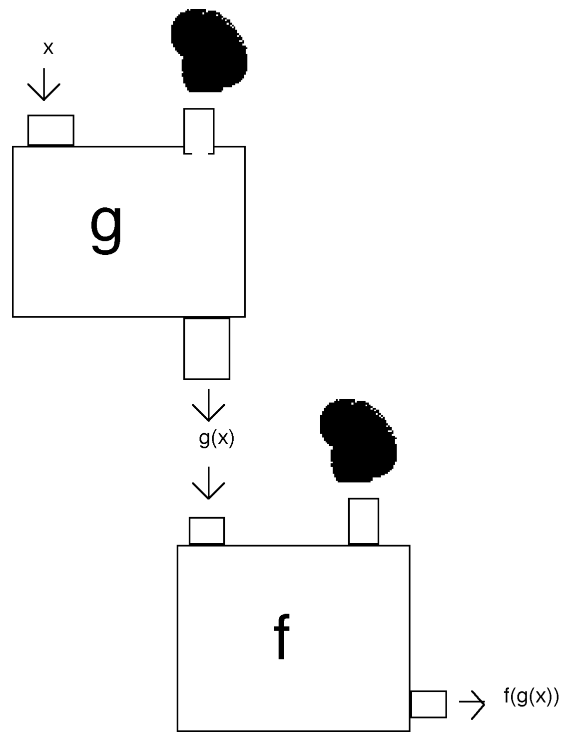

Figure 1.9: To find $(f \circ g)(x)$, first put x into the g machine, then into the f machine. Look at that pollution. Do you think this is adding to the level of carbon dioxide in the atmosphere?

The domain of $(f \circ g)(x)$ consists of real numbers that are in the domain of g (since you apply g to x first), and we also require that $g(x)$ lies in the domain of f (since we apply f to $g(x)$).

Example. Let $f(x) = x^2$ and $g(x) = 1/x$, find $(f \circ g)(x)$, and find the domain of $(f \circ g)(x)$. Well, $(f \circ g)(x) = f(g(x))$. First we simplify $g(x) = 1/x$. So $f(g(x)) = f(1/x)$. Now $f(1/x)$ is just like $f(x)$, except that we replace x with $1/x$. Thus $f(1/x) = (1/x)^2$. Hence $(f \circ g)(x) = (1/x)^2$. To find the domain of this function, we have to check the domain of g, which is $\{x|x \neq 0\}$. The range of g is all real numbers except 0, and the domain of f is all real numbers. Thus f will accept any value of $g(x)$. It follows that the domain of $(f \circ g)(x)$ is the same as the domain of $g = \{x|x \neq 0\}$.

Example. Let $f(x) = \sqrt{x^2+1}$, and let $g(x) = \sin x$. Find $(f \circ g)(x)$, $(g \circ f)(x)$, and their respective domains.

$(f \circ g)(x) = f(g(x)) = f(\sin x) = \sqrt{(\sin x)^2 + 1}$. Since the domain of $\sin x$ consists of all real numbers, and since $(\sin x)^2 + 1$ is always non-negative, the domain of $(f \circ g)(x)$ consists of all real numbers

$(g \circ f)(x) = g(f(x)) = g(\sqrt{x^2+1}) = \sin(\sqrt{x^2+1})$. Since the domain of $x^2 + 1$ is all real numbers and since the domain of $\sin x$ is all real numbers, the domain of $(g \circ f)(x)$ is all real numbers.

Example. Let $f(x) = 6$, and let $g(x) = x^3$. Find $(f \circ g)(x)$ and $(g \circ f)(x)$. (Note that since both of these functions are defined for all real numbers, each composite has as domain all real numbers).

$$(f \circ g)(x) = f(g(x)) = f(x^3) = 6$$
$$(g \circ f)(x) = g(f(x)) = g(6) = 6^3 = 216$$

As we see in the examples above, in general, $(f \circ g)(x) \neq (g \circ f)(x)$.