5.3 The Fundamental Theorem of Calculus

You might remember that in mathematics a theorem is a statement that can be proven true. A good theorem relates ideas that do not seem to be related. The fundamental theorem of calculus relates differentiation and integration. It's an extremely good theorem.

You saw in lab that $\int_0^x 3t\,dt = \frac{3x^2}{2}$, and later you computed that $\frac{d}{dx}\frac{3x^2}{2} = 3x$. Similarly, $\int_2^x (-t+9)dt = (1/2)(16-x)(x-2)$, and $\frac{d}{dx}(1/2)(16-x)(x-2) = -x+9$. These computations were not coincidences.

These examples are consistent with what is proven by the
Fundamental Thoerem of Calculus, Part I If f is a continuous function on $[a,b]$, then $g(x) = \int_a^x f(t)dt$ is continuous on $[a,b]$ and differentiable over the interval (a,b), with $\frac{dg}{dx} = \frac{d}{dx}\int_a^x f(t)dt = f(x)$ for $a < x < b$. In other words, g is an anti-derivative of f.

Sketch of Proof: By definition $\frac{dg}{dx} = \lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$. We'll try to work with this to get it equal to $f(x)$. First, we note that $g(x+h) - g(x) = \int_a^{x+h} f(t)dt - \int_a^x f(t)dt = \int_x^{x+h} f(t)dt$. (This is one of the properties of integrals discussed in a previous section.) Using this, we get $\frac{dg}{dx} = \lim_{h\to 0}\frac{\int_x^{x+h} f(t)dt}{h} \approx \lim_{h\to 0}\frac{f(x)\cdot h}{h} = f(x)$. To see intuitive justification of the $\approx$, see figure 5.1.

<u>Examples:</u>

$$\frac{d}{dx}\int_0^x 3t\,dt = 3x$$

$$\frac{d}{dx}\int_2^x (-t+9)dt = -x+9$$

$$\frac{d}{dx}\int_0^x (t^3+t^2+1)dt = x^3+x^2+1$$

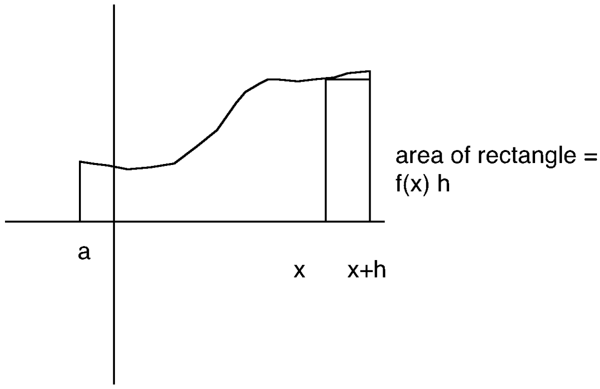

Figure 5.1: This figure suggests why $\lim_{h \to 0} \frac{\int_x^{x+h} f(t)dt}{h} \approx \lim_{h \to 0} \frac{f(x) \cdot h}{h}$.

$$\frac{d}{dx} \int_7^x \frac{1}{t} dt = \frac{1}{x}$$

<u>Example:</u> Find $\frac{dy}{dx}$ if $y = \int_1^{x^2} \cos t dt$

Remember the chain rule: $\frac{d}{dx} g(f(x)) = g'(f(x))f'(x)$. In this case, $f(x) = x^2$, and $g(u) = \cos u$. So we get $\frac{dy}{dx} = \cos(x^2)(2x)$.

<u>Example:</u> Find $\frac{d}{dx} \int_1^{\sqrt{x}} \sin t dt$

It's $(\sin(\sqrt{x})) \left(\frac{1}{2\sqrt{x}} \right)$.

Now we'll introduce the fundamental theorem of calculus, part II:

The Fudamental Theorem of Calculus, Part II: If f is continuous on the closed interval $[a, b]$, then $\int_a^b f(t)dt = F(b) - F(a)$, where $F(x)$ is any function with the property that $F'(x) = f(x)$.

In practice, this is an extremely useful theorem when it comes to evaluating definite integrals. Now we'll give a proof of this theorem:

Let $g(x) = \int_a^x f(t)dt$. By the Fundamental Theorem of Calculus, I, $g'(x) = f(x)$. Now consider a function $F(x)$ with the property that $F'(x) = f(x)$. By one of the corollaries of the Mean Value Theorem, $F(x)$ and $g(x)$ differ by a constant. That is to say, $F(x) = g(x) + C$. Now, $F(b) - F(a) = (g(b)+C) - (g(a)+C) = g(b) - g(a)$. By definition, $g(b) - g(a) = \int_a^b f(t)dt - \int_a^a f(t)dt = \int_a^b f(t)dt$ because $\int_a^a f(t)dt = 0$. Done!

Now we can compute a lot of definite integrals with ease and enjoyment:

Example: $\int_0^4 (6x+4)dx$

Now all we need is a function whose derivative is $6x + 4$. Try $3x^2 + 4x$. It works. So $\int_0^4 (6x+4)dx = (3x^2 + 4x|_0^4 = 64 - 0 = 64$.

Example: $\int_0^\pi \sin x dx$

This time we need a function $F(x)$ with $F'(x) = \sin x$. Try $-\cos x$. Thus $\int_0^\pi \sin x dx = (-\cos x|_0^\pi = -(1) - (-1) = 2$.

In the next section, we'll go over some basic anti-differentiation formulas.

5.4 Anti-Derivatives and Indefinite Integrals

OK, so finding anti-derivatives is a good thing to do. Let's think about this some more. A function F is an antiderivative of the function f on the interval I if $F' = f$ for all values of x in I.

We denote this relationship by $\int f(x)dx = F(x) + C$. It's called "the indefinite integral of $f(x)$." This is because the Fundamental Theorem of

Calculus so closely relates the definite integral of a function and the anti-derivatives of a function.

Here are some common anti-derivatives:

$\int x^n dx = \frac{x^{n+1}}{n+1} + C$ provided $n \neq -1$ (what happens if you try to apply this formula with $n = 1$?) This formula comes from the power rule of differentiation.

$$\int \sin x dx = -\cos x + C$$

$$\int \cos x dx = \sin x + C$$

$$\int k dx = kx + C$$

$$\int \sec^2 x dx = \tan x + C$$

There are a few other nice formulas listed in Stewart, p. 347.

Example: Find $\int (\sin x + x^2) dx$

Just like the derivative of the sum of two functions is the sum of the derivative, so is the anti-derivative of the sum of two functions the sum of the two anti-derivatives (although we'll only need one constant). Thus, $\int (\sin x + x^2) dx = -\cos x + \frac{x^3}{3} + C$.

Example: A ball is thrown up with an initial velocity of 16 feet per second. When does the ball reach its maximum height? How high does the ball travel? You may assume that acceleration due to gravity is 32 feet per second squared.

We'll make the up direction positive, the down direction negative. Since gravity pulls down, we have $a = -32$. Since the derivative of velocity is acceleration, we have $\int -32 dt =$ velocity (up to adding a constant). So, $v(t) = -32t + C$. We'll use the fact that $v(0) = 16$ to find C. $v(0) =$

$-32(0) + C = 16$, so $C = 16$. It follows that $v(t) = -32t + 16$. The ball reaches it's maximum height at the instant where velocity changes from positive to negative, so $v(t) = -32t + 16 = 0$, using some algebra, we get $t = .5$ seconds.

Now we'll find out how high the ball travelled. We'll assume that $s(0) = 5$, which seems like a reasonable assumption since most people are over 5 feet tall. Now, since $v(t) = -32t + 16$, then $s(t) = -16t^2 + 16t + C$. Since $s(0) = 5 = C$, we have that $C = 5$, and $s(t) = -16t^2 + 16t + 5$. Now we'll find how high up the ball travelled by finding $s(.5) = -16(.5)^2 + 16(.5) + 5 = -4 + 8 + 5 = 9$. So the ball travelled up in the air 9 feet.

5.5 Substitution in integration

Unfortunately, it is difficult to find anti-derivatives for many functions. In calc II, you will spend a chapter on techniques for finding anti-derivatives. We'll end calc I with one such technique. First, you need to remember the chain rule: $\frac{d}{dx}(f(g(x)) = f'(g(x))g'(x)$. Which says the derivative of a composition equals the derivative of the outside function times the derivative of the inside. We're going to run this rule backwards to find anti-derivatives. So, we know that $\int f'(g(x))g'(x)dx = f(g(x)) + C$, since $\frac{d}{dx}(f(g(x)) = f'(g(x))g'(x)$.

So, if we start with an integral of the form: $\int f(g(x))g'(x)dx$, we'll do the following:

1. Substitute $u = g(x)$. You pick what you want to set u equal to. This is a bit of an art. If at first you do not succeed, try, try again..... You want to pick u so that the integral becomes simpler, and so that du looks like other terms in the integral. Once you have picked u, differentiate to find du, then switch the integral over to u and du. There shouldn't be any x's involved in the new integral.

2. Anti-differentiate.

3. Replace u by $g(x)$ so that your final answer is in terms of x.

Example: $\int (x^7+1)^{99} 7x^6 dx$

This time try $u = x^7 + 1$. Then $du = 7x^6 dx$. The integral becomes: $\int u^{99} du = \frac{u^{100}}{100} + C$. Now plug back in for x, we get $\frac{(x^7+1)^{100}}{100} + C$.

Example: $\int \sin(7x) dx$

This time try $u = 7x$. Then $du = 7dx$. The integral becomes: $\int \sin(u) dx$. Whoah.... there's a dx in there. We can't have that. Can we write dx in terms of u and du? Sure we can. We have the equation $du = 7dx$, solve it for dx to get $dx = \frac{du}{7}$. Our integral is now: $\int \sin(u)\frac{du}{7} = \frac{1}{7}\int \sin u du = \frac{-\cos u}{7} + C = \frac{-\cos 7x}{7} + C$

5.5.1 Substitution in Definite Integrals:

If we make the substitution $u = g(x)$, we get: $\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$. Notice we also change the limits of integration to reflect how u varies as x varies from a to b.

Example: $\int_{-1}^{1} (x^2+1)^{\frac{1}{2}} 2x dx$

I hope you would pick $u = x^2 + 1$. Then $du = 2xdx$. So the integral becomes $\int_?^? u^{\frac{1}{2}} du$. But what are the new limits? When $x = -1, u = (-1)^2 + 1 = 2$, so the lower limit is 2. When $x = 1$, $u = (1)^2 + 1 = 2$, so the upper limit is also 2. Thus the new integral is: $\int_2^2 u^{\frac{1}{2}} du = 0$.

Example: $\int_0^{\pi/4} \tan x \sec^2 x dx$.

Pick $u = \tan x$. So $du = \sec^2 x dx$. When $x = 0$, $u = \tan(0) = 0$, and when $x = \pi/4$, $u = \tan(\pi/4) = 1$. Thus the new integral is $\int_0^1 u du = \left(\frac{u^2}{2}\right)_0^1 = 1/2$.