## 5.2 Definite Integrals and Riemann Sums, Part II

Here we modify slightly the definition given in the previous section:

If $f$ is continuous for $a \leq x \leq b$, define $\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{\infty} f(x_i^*)\Delta x$. This number is called the definite integral of $f(x)$ from $x = a$ to $x = b$, where $\Delta x = \dfrac{b-a}{n}$, $x_i = a + i\Delta x$, and $x_i^*$ is any number in the interval $[x_{i-1}, x_i]$. One can prove, assuming that $f$ is continuous, that the value of the definite integral is independent of the choice of $x_i^*$. In particular, one could choose left endpoints or right endpoints, or midpoints, etc...

Your text points out a few useful observations. We will repeat them here:

1. The integral sign comes from the letter S, which stands for sum.

2. $\int_a^b f(x)dx$ is a number. We also have that
   $$\int_a^b f(x)dx = \int_a^b f(t)dt = \int_a^b f(r)dr = \int_a^b f(Fred)d(Fred)$$

3. Assuming $f$ is continuous, one can prove that the above limit exists. One can also prove, assuming $f$ continuous, that the value of the definite integral is independent of the choice of $x_i^*$. In particular, one could choose left endpoints or right endpoints, or midpoints, etc... We'll spare you the proof here, it's the kind of topic that might be taken up in an advanced calculus course.

4. If $f(x) \geq 0$ for $a \leq x \leq b$, then $\int_a^b f(x)dx =$ area between the graph of $y = f(x)$ and the $x$ axis. If $f(x)$ is not always greater than or equal to zero, then $\int_a^b f(x)dx =$ area above the x axis (and below $y = f(x)$) minus area below x-axis (and above $y = f(x)$).

### 5.2.1 Properties of the Definite Integral

Here we discuss some nice properties of the definite integral. Many have geometric interpretations. We will not prove the properties rigorously, but we will give some justification.

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

This is true because for the second integral, $\Delta x = (a-b)n = -(b-a)/n$, which is $\Delta x$ for the first integral.

$$\int_a^a f(x)dx = 0$$

In this case, we are integrating over a region with no width. So $\Delta x = (a-a)/n = 0$.

$$\int_a^b cdx = c(b-a)$$

Here $c$ is a constant. If $c$ is positive, the integral gives the area of a rectangle whose base is of length $b-a$ and whose height is $c$.

The next couple of properties are analogous to the sum rule, the difference rule, and the constant multiple rule of differentiation:

$$\int_a^b (f(x)+g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

$$\int_a^b (f(x)-g(x))dx = \int_a^b f(x)dx - \int_a^b g(x)dx$$

$$\int_a^b cf(x)dx = c\int_a^b f(x)dx$$

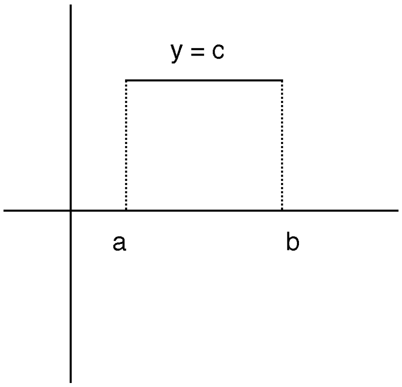

Figure 5.1: The integral $\int_a^b c dx = c(b-a)$.

The next property can be informally justified using a picture, for the case where $f(x) \geq 0$ (see Figure 5.2).

$$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$$

One can always use area interpretations as a short cut for computing definite integrals, as the next two examples show:

<u>Example:</u> Compute $\int_0^1 2x dx$.

This definite integral gives the area of a triangle with base length 1 and height 2. The area is $(1/2)(1)(2) = 1$.

<u>Example:</u> $\int_0^3 \sqrt{9-x^2}dx$

This definite integral gives the area of a quarter of a circle of radius three. The area is $(1/4)\pi(3)^2 = \frac{9\pi}{4}$.

The last property we shall discuss is Stewart's property 8. Again, we'll use a picture to informally justify it's validity when $f(x) \geq 0$.

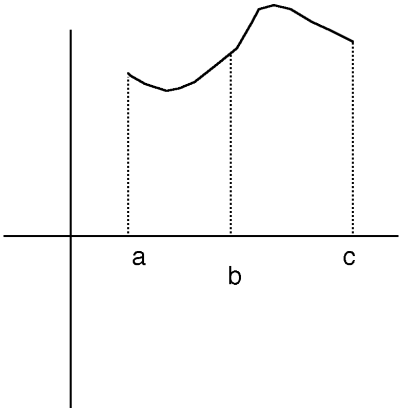

Figure 5.2: $\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$

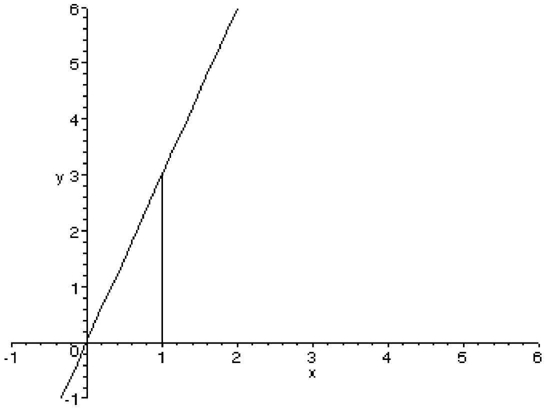

Figure 5.3: $\int_0^1 2xdx = 1$

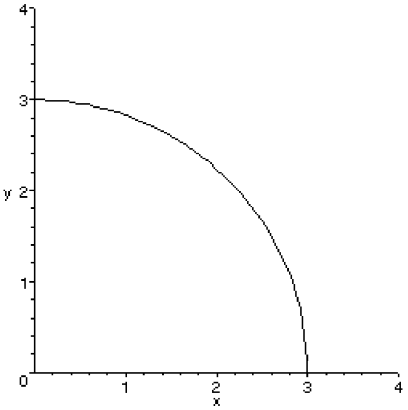

Figure 5.4: $\int_0^3 \sqrt{9-x^2}dx = \dfrac{9\pi}{4}$

Property 8: Let $f(x)$ be continuous for $a \leq x \leq b$, and let $m$ represent the minimum value that $f(x)$ takes on over $[a,b]$, and let $M$ represent the maximum value that $f(x)$ takes on over $[a,b]$, then:

$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$$

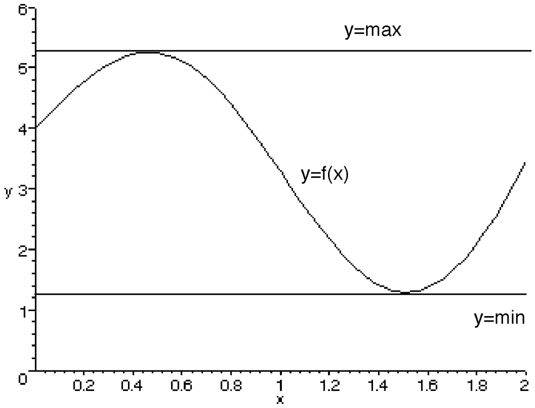

Figure 5.5: $\int_0^1 2x dx = 1$