

4.4 Limits at Infinity and Horizontal Asymptotes

Note that in this section, we will not discuss derivatives. The topics discussed will help us out when we discuss curve sketching later in the chapter.

We start with some informal definitions. More formal definitions are possible; they would involve ϵ and N . You will/would see such definitions in our advanced calculus course.

Let f be defined on the interval (a, ∞) for some number a . Then $\lim_{x \rightarrow \infty} f(x) = L$ means that $f(x)$ can be made arbitrarily close to L provided x is sufficiently large. We say “the limit of $f(x)$ as x approaches infinity is L .”

Similarly: Let f be defined on the interval $(-\infty, a)$ for some number a . Then $\lim_{x \rightarrow -\infty} f(x) = L$ means that $f(x)$ can be made arbitrarily close to L provided x is sufficiently large negative. We say “the limit of $f(x)$ as x approaches negative infinity is L .”

Example: $\lim_{x \rightarrow \infty} \frac{1}{x} = 0$ because we can make $1/x$ arbitrarily close to 0 provided x is sufficiently large. Similarly, $\lim_{x \rightarrow -\infty} \frac{1}{x} = 0$.

If either $\lim_{x \rightarrow \infty} f(x) = L$ or $\lim_{x \rightarrow -\infty} f(x) = L$ then $y = L$ is a **horizontal asymptote** of the curve $y = f(x)$.

So, from the previous example, we know that $y = 0$ is an asymptote of the curve $y = 1/x$.

Limits of Rational Functions

Example: Find $\lim_{x \rightarrow \infty} \frac{2x^2 + 6}{7x^2 + 2x + 1}$.

Key step: divide each term by x to the highest power in the denominator.

$$\lim_{x \rightarrow \infty} \left(\frac{2x^2 + 6}{7x^2 + 2x + 1} \right) \left(\frac{1/x^2}{1/x^2} \right).$$

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$$= \frac{2 + 6/x^2}{7 + 2/x + 1/x^2} = 2/7.$$

Example: Find all horizontal and vertical asymptotes of $y = \frac{2x^2 + 6}{2x^3 + 2}$.

$$\begin{aligned} \lim_{x \rightarrow -\infty} \frac{2x^2 + 6}{2x^3 + 2} &= \lim_{x \rightarrow -\infty} \left(\frac{2x^2 + 6}{2x^3 + 2} \right) \left(\frac{1/x^3}{1/x^3} \right) \\ &= \lim_{x \rightarrow -\infty} \frac{2/x + 6/x^3}{2 + 2/x^3} = 0/2 = 0. \end{aligned}$$

Similarly, $\lim_{x \rightarrow \infty} \frac{2x^2 + 6}{2x^3 + 2} = 0$. So $y = 0$ is the only horizontal asymptote.

A vertical asymptote may occur when the denominator is 0. So we need to check when $2x^3 + 2 = 0$. In other words (algebra...) $x^3 = -1$, so $x = -1$.

We need to check:

$$\lim_{x \rightarrow -1^+} \frac{2x^2 + 6}{2x^3 + 2} = \frac{8}{0^+} = +\infty.$$

$$\text{Also: } \lim_{x \rightarrow -1^-} \frac{2x^2 + 6}{2x^3 + 2} = \frac{8}{0^-} = -\infty.$$

So $x = -1$ is a vertical asymptote. See figure 4.1.

Example: Find $\lim_{x \rightarrow \infty} (\sqrt{x} - \sqrt{x-1})$.

Since both \sqrt{x} and $\sqrt{x-1}$ both go to ∞ , we cannot use any limit laws to simplify. We can use the standard trick of multiplying both numerator and denominator by $\sqrt{x} + \sqrt{x-1}$. This will help to clear out some square roots.

$$\begin{aligned} \lim_{x \rightarrow \infty} (\sqrt{x} - \sqrt{x-1}) &= \lim_{x \rightarrow \infty} (\sqrt{x} - \sqrt{x-1}) \left(\frac{\sqrt{x} + \sqrt{x-1}}{\sqrt{x} + \sqrt{x-1}} \right) \\ &= \lim_{x \rightarrow \infty} \frac{x - (x-1)}{\sqrt{x} + \sqrt{x-1}} \\ &= \lim_{x \rightarrow \infty} \frac{1}{\sqrt{x} + \sqrt{x-1}} = 0 \end{aligned}$$

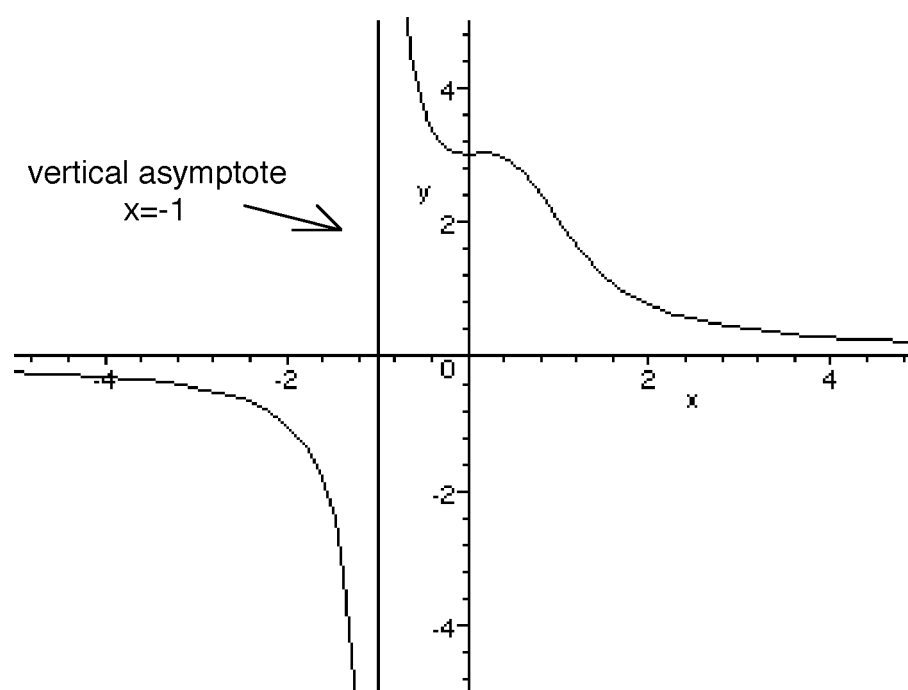


Figure 4.1: The graph of $y = \frac{2x^2 + 6}{2x^3 + 2}$.

Example: This one should be intuitively obvious: $\lim_{x \rightarrow \infty} x^2 = \infty$.

Example: Find $\lim_{x \rightarrow \infty} x^2 - x$.

Again, since both x^2 and x both go to infinity, we cannot just subtract their individual limits. We can, however, factor $x^2 - x = x(x - 1)$. Now, since both x and $x - 1$ both go to infinity, their product will also go to infinity. The limit is ∞ .

4.5 Summary of Curve Sketching

The point of this section is to review how to sketch a curve, given the equation of a function. You will want to use your graphing calculator to check your work. Sketching curves by hand does have some advantages over using just the graphing calculator to sketch. You can find *exact* values of maxima and minima. In addition, graphing calculators can present ambiguous pictures of graphs. Often, a graph can look flat in places where it is actually increasing or decreasing (see figure 4 on p. 264 of Stewart).

As done in Stewart, we go through the guidelines for sketching a curve here.

1. Domain. This tells you what x values you need.
2. Intercepts. Find the value of $f(0)$. If possible, find the values of x that give $f(x) = 0$.
3. Symmetry. Check if the function is even (does $f(-x) = f(x)$)? If so, you only need to figure out what the curve looks like for $x \geq 0$. Then reflect across the y axis. If the function is odd $f(-x) = -f(x)$, then still you only need to figure out what the curve looks like for $x \geq 0$. Once you know this, you can rotate the graph 180 degrees.

4. Asymptotes. Determine horizontal asymptotes, if they exist (see previous section for review). To find the vertical asymptotes of a rational function, check where the denominator is 0 after cancelling common factors.
5. Find **intervals of increase and decrease** using the first derivative.
6. Find **local maxima and minima** using the first or second derivative test. Find the points at which the maxima and minima occur, and use the points in your sketch.
7. Concavity and points of inflection. Find them using the second derivative test. Plot points of inflection before you sketch the curve.
8. Sketch away!

Example: Sketch $y = x^3 - x$.

1. Domain is all reals.
2. $f(0) = 0$, and $x^3 - x = x(x^2 - 1) = 0$ means either $x = 0$ or $x = \pm 1$. These are the intercepts. So the points $(0, 0)$, $(-1, 0)$, and $(1, 0)$ lie on the graph.
3. The function is odd, since $f(-x) = (-x)^3 - (-x) = -x^3 + x = -(x^3 - x) = -f(x)$.
4. This is a polynomial. It has no asymptotes.
5. $f'(x) = 3x^2 - 1$. This is zero when $x = \pm\sqrt{1/3}$. By checking test values, we find that f is increasing on $(-\infty, -\sqrt{1/3})$ and on $(\sqrt{1/3}, \infty)$, and decreasing on $(-\sqrt{1/3}, \sqrt{1/3})$.

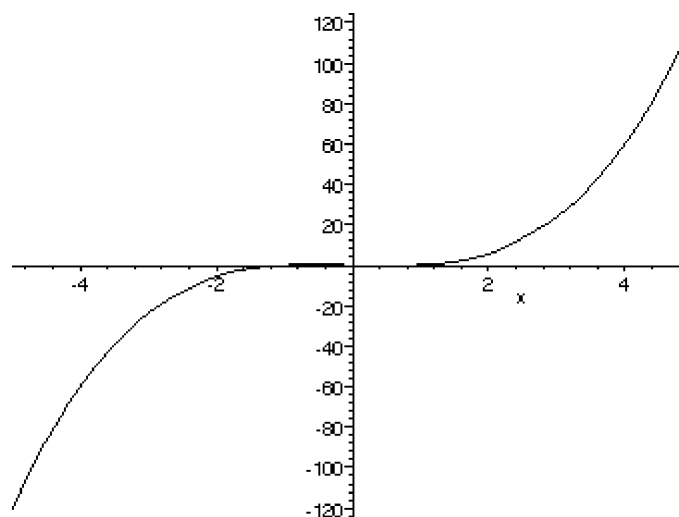


Figure 4.2: The graph of $y = x^3 - x$. This scale is deceiving.

6. By the first derivative test, there is a local max at $(-\sqrt{1/3}, (2/3)\sqrt{1/3})$, and there is a local min at $(\sqrt{1/3}, -(2/3)\sqrt{1/3})$
7. $f''(x) = 6x$. This shows that the graph of $y = f(x)$ is concave up for x in the interval $(0, \infty)$, and concave down for x in the interval $(-\infty, 0)$.
8. As I was making up the graph of this function, I cheated, and had a program, Maple, do the plot. The first graph it gave me was deceptive. It didn't look like the graph had any local maxs or mins. See figure 4.2.

I knew, however, from the steps outlined above, that the function had both a local max and a local min. I then reset the scale and had maple graph again. The better result is shown in figure 4.3.

Example: Graph $f(x) = \frac{x+4}{x-2}$.

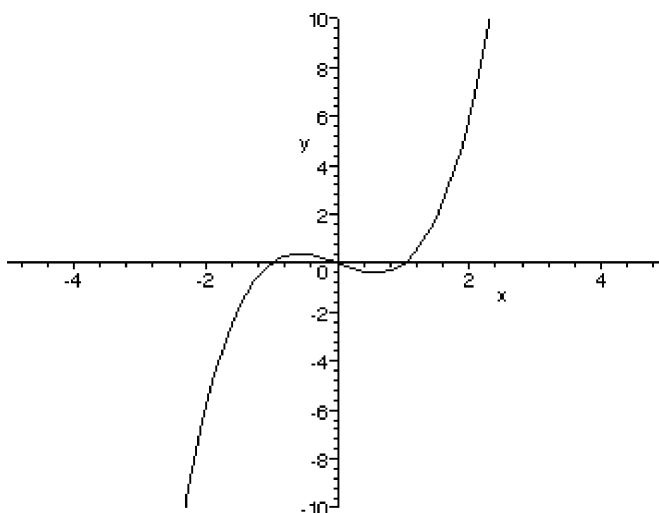


Figure 4.3: The graph of $y = x^3 - x$. This scale is improved.

1. The domain is all real numbers except $x = 2$.
2. $f(0) = -2$. The y intercept can be found by setting the numerator equal to 0. They y intercept is -4 .
3. There is no even or odd symmetry.
4. Check $\lim_{x \rightarrow \infty} \frac{x+4}{x-1} = 1$. Similarly, $\lim_{x \rightarrow -\infty} \frac{x+4}{x-1} = 1$

To find the vertical asymptote(s), check values of x that make the denominator 0. We have that $\lim_{x \rightarrow 2^+} \frac{x+4}{x-2} = \frac{6}{0^+} = \infty$. Similarly,

$$\lim_{x \rightarrow 2^-} \frac{x+4}{x-2} = \frac{6}{0^-} = -\infty.$$

5. $f'(x) = \frac{-6}{(x-2)^2}$. This is less than zero for all x , except at $x = 2$, where the quotient is undefined. The function is decreasing at all real numbers where it is defined.

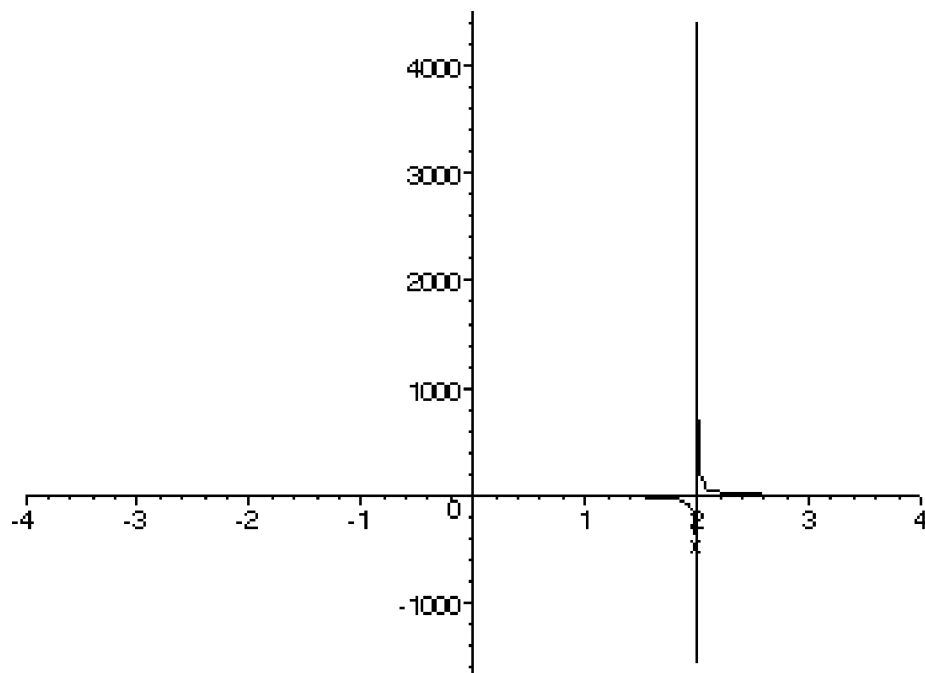


Figure 4.4: The graph of $y = \frac{x+4}{x-2}$. Bad scale.

6. There is no local max or local min.
7. $f''(x) = 12(x-2)^{-3}$. This is undefined at $x = 2$. Since $f''(0) = -96$, we may conclude that $f''(x) < 0$ for $x < 2$. Since $f''(3) = 12$, we may conclude that $f''(x) > 0$ for $x > 2$. So the function is concave down on the interval $(-\infty, 2)$ and concave up on the interval $(2, \infty)$.
8. Sketch away. . . . See figure 4.4.

Again, this time Maple gave me a bad scale (see figure 4.4). You can really even tell if the horizontal asymptote is at $y = 1$ or not. A second try gives a nicer picture (see Figure 4.5).

4.8 Optimization Problems

The focus of this section is one applications of calculus to practical problems, in other words, word problems. Here are some steps you can take to try to solve these problems:

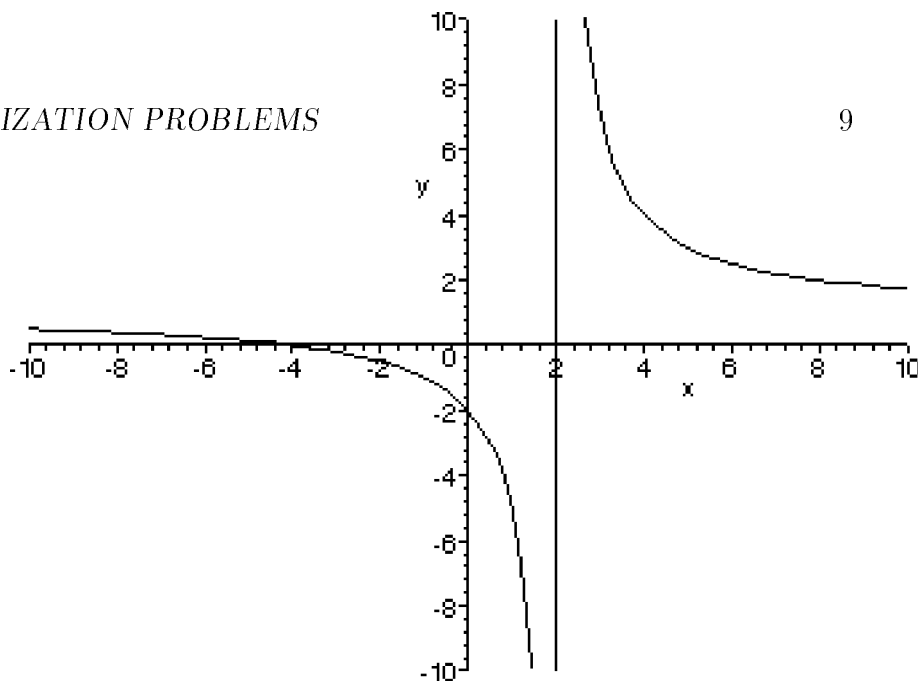


Figure 4.5: The graph of $y = \frac{x + 4}{x - 2}$. Better scale.

1. Read the question carefully.
2. Draw a picture, if it's appropriate.
3. Make variables, and identify the quantity you are trying to maximize or minimize.
4. Make equations. Get a formula to express the quantity you are trying to minimize/maximize in terms of just one other variable.
5. If the quantity you want to maximize/minimize is of the form $f(x)$, then you want to look at extreme values of x , and values of x that make $f'(x)$ equal to 0 or undefined.

Example: A farmer has 500 feet of fence and wants to fence a rectangular field that borders a river. What dimensions give the largest area.

This one's a classic. First draw a picture. You should get a picture like the one in figure 4.6.

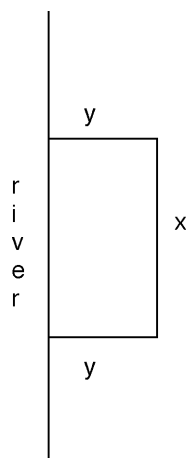


Figure 4.6: A rectangular fence, and a river.

From the picture, and the wording of the question, we can get two equations: the perimeter is $2y + x = 500$, and the area is $A = xy$. We're trying to maximize area. We'll use the first equation to solve for $x = 500 - 2y$. Now we'll substitute this into the second equation: $A = (500 - 2y)y = 500y - 2y^2$, and using common sense, we know that $0 < y < 250$. To find the maximum, we'll use the derivative: $A'(y) = 500 - 4y$. This is 0 when $y = 500/4 = 125$. Since $A'(y) > 0$ for $y < 125$, and $A'(y) < 0$ for $y > 125$, we must be at a global max. So, pick $y = 125$, and $x = 250$.

Example: Find 2 non-negative numbers whose sum is 20 and whose product is as large as possible.

OK, so again we get two equations: $x + y = 20$, and $P = xy$. We want to maximize the product, so use $y = 20 - x$ to get $P = x(20 - x)$, with $0 \leq x \leq 20$ (why is x confined as such?)

So $P(x) = 20x - x^2$. Then $P'(x) = 20 - 2x$. This is 0 when $x = 10$. Note that $P'(x) > 0$ for all $x < 10$ and $P'(x) < 0$ for all $x > 10$. This indicates that $x = 10$ gives a local maximum value (see box on p. 280). Thus $x = 10$ and $y = 20 - x = 10$ and the maximum product is $xy = (10)(10) = 100$.