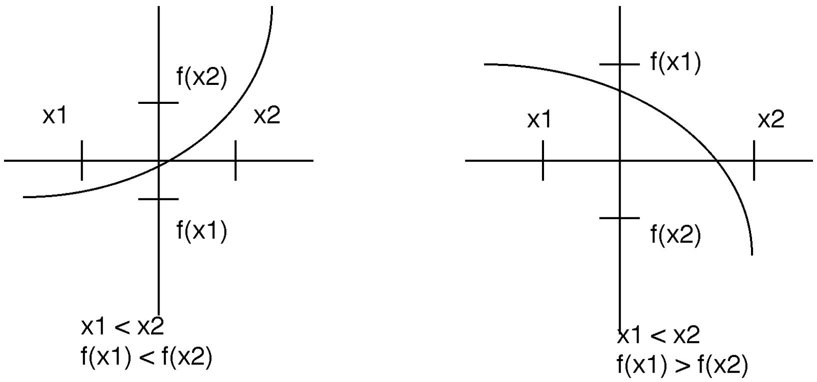

Figure 4.1: The first function is increasing, the second is decreasing.

4.3 Derivatives and the shape of a graph

The first derivative is going to tell us when a differentiable function is increasing and when it is decreasing. We need to define carefully what it means for a function to be increasing.

A function $f(x)$ is **increasing** on an interval if for any choice of x_1 and x_2 in the interval, with $x_1 < x_2$, we have $f(x_1) < f(x_2)$. Similarly, a function $f(x)$ is **decreasing** on an interval if for any choice of x_1 and x_2 in the interval , with $x_1 < x_2$, we have $f(x_1) > f(x_2)$. A good way to understand this definition is by examining a picture (see Figure 4.1).

Now we can state the increasing/decreasing test. It's validity rests on the foundation of the Mean Value Theorem. **The increasing/decreasing test:** If $f'(x) > 0$ for all x in a given interval, then f is increasing on that interval. If $f'(x) < 0$ for all x in a given interval, then f is decreasing on that interval.

Here's a proof for the increasing case. The decreasing case is similar. Let $x_1 < x_2$ lie in the given interval, and suppose that $f'(x) > 0$ for all x in the interval. By the Mean Value Theorem, there is a number c in the interval

(x_1, x_2) such that $f'(c) = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$. By clearing fractions, this means $f'(c)(x_2 - x_1) = f(x_2) - f(x_1)$. By assumption, $f'(c) > 0$, and since $x_2 > x_1$, it follows that $x_2 - x_1 > 0$. We may conclude that $f(x_2) - f(x_1) > 0$, thus that $f(x_2) > f(x_1)$.

Since the reasoning given above works for any x_1 and x_2 in the interval with $x_1 < x_2$, we may conclude that f is increasing on that interval. This concludes our proof.

Example: Find intervals where $f(x) = x^3 - 3x$ is increasing and intervals where $f(x)$ is decreasing.

We won't give a graph here (we don't need to), but you should check our work later by graphing on your calculator. All we have to do is find $f'(x)$, and figure out where it's negative and positive. $f'(x) = 3x^2 - 3$. We'll find roots first: $3x^2 - 3 = 0$ means that $x^2 = 1$, so $x = \pm 1$. Since $3x^2 - 3$ is a continuous function, by the Intermediate Value Theorem, it cannot change sign over the following three intervals: $(-\infty, -1)$, $(-1, 1)$, and $(1, +\infty)$. We can use test values from each of these intervals to determine the sign of $f'(x)$. Let's see… $f'(-2) = 9$, so f' is positive on the interval $(-\infty, -1)$. We have $f'(0) = -3$, so f' is negative over $(-1, 1)$. Finally, $f(2) = 9$, so f' is positive over $(1, +\infty)$. You might want to make a number line as in figure 4.2 and indicate the sign of f' over intervals by $+$ and $-$. Finally, we may conclude that f is increasing on the intervals $(-\infty, -1)$ and $(1, +\infty)$, and f is decreasing on the interval $(-1, 1)$. At this point, you might want to check our findings with a graph.

You may have noticed the following was true in the previous example, it also holds true in general:

The first derivative test: Suppose c is a critical number of a continuous function f. (do you remember the definition of continuous? If not, you should look it up.)

1. If f' changes from $+$ to $-$ at c, then f has a local max.
2. If f' changes from $-$ to $+$ at c, then f has a local min.

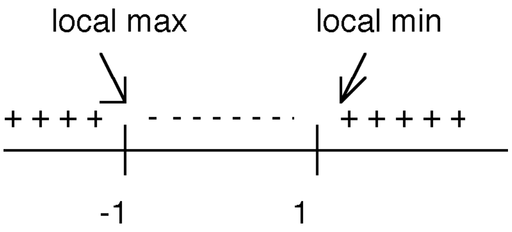

Figure 4.2: The number line indicates intervals where f is increasing and intervals where f is decreasing.

3. If f' does not change sign at c, then f has no local max or min at c.

4.3.1 Concavity

The second derivative also tells us a lot about the graph of a differentiable function. First, we define concavity:

The graph of a differentiable function $y = f(x)$ is **concave up** if and only if y' is increasing, and the graph of $y = f(x)$ is **concave down** if y' is decreasing (see figure 4.3).

By the increasing/decreasing test, and assuming our function can be twice differentiated, y' is increasing if and only if $y'' > 0$, and y' is decreasing if and only if $y'' < 0$. It follows that $y = f(x)$ is concave up if $f''(x) > 0$, and concave down if $f''(x) < 0$.

Example: Where is $f(x) = x^3 - 3x$ concave up? down?

$f'(x) = 3x^2 - 3$
$f''(x) = 6x$. This is 0 when $x = 0$. Since $f''(-1) = -6 < 0$, we may conclude that $f''(x) < 0$ for x in $(-\infty, 0)$. Since $f''(1) = 6 > 0$, we may conclude that $f''(x) > 0$ for x in $(0, +\infty)$. So, $y = f(x)$ is concave up for x in $(0, +\infty)$, and

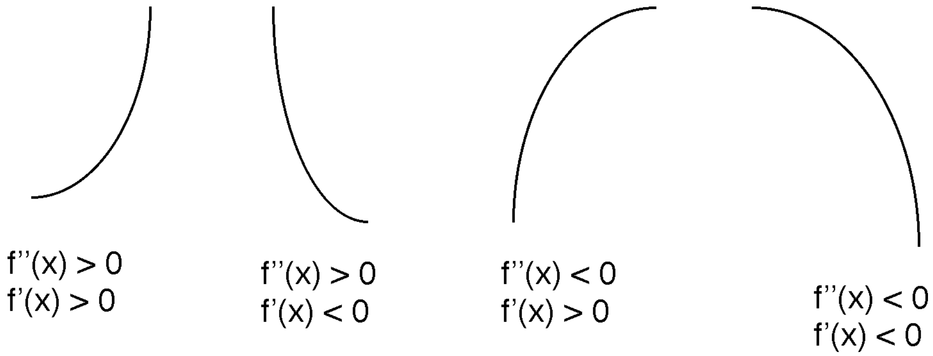

Figure 4.3: The four possible shapes of a piece of a graph. If $f'(x) > 0$, the graph is increasing. If $f''(x) > 0$, the graph is concave up.

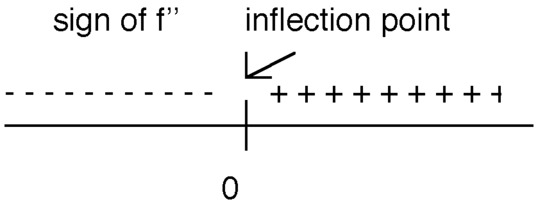

Figure 4.4: The function $f(x) = x^3 - 3x$ has an inflection point at $x = 0$.

it is concave down for x in $(-\infty, 0)$. Again, you should check you calculator's graph.

A point on the graph of a curve where the concavity changes is called and **inflection point**. In the example above $f(x) = x^3 - 3$, 0 is an inflection point. To find an inflection point of a twice differentiable function, a good place to start is to find where $f''(x) = 0$, then check to see if the sign of $f''(x)$ changes.

Example: Find all inflection points of $y = x^4$.

$y' = 4x^3$, so $y'' = 12x^2$. So $y'' = 0$ only when $x = 0$. Let's check some sample points: $f''(-1) = 12$, so $f''(x) > 0$ for x in the interval $(-\infty, 0$, and $f''(1) = 12 > 0$, so $f''(x) > 0$ for x in the interval $(0, +\infty)$. It follows that this function has no inflection points, even though its second derivative is 0 when $x = 0$.