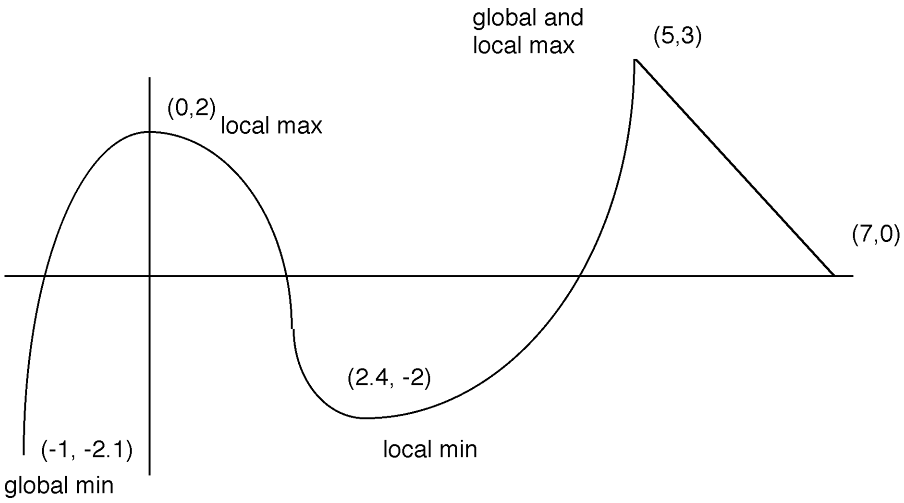

Figure 4.1: The graph of a function $y = f(x)$ over the domain $[-1, 7]$.

4.1 Maxs and Mins

An important application of differentiation involves finding maximum and minimum values of functions. Perhaps your function is cost, and you want to minimize it. The geometric shape of a soap bubble is determined by the tendency of the soap bubble to minimize its surface area, given that it is enclosing a fixed volume of air. Essentially sophisticated calculus is all you need to predict the shape of soap bubbles.

Most students find the concept of maxs and mins to be fairly intuitive. If you look at figure 4.1 you can see some examples. Here are some definitions:
A function f has an **absolute (or global) maximum** at $x = c$ provided $f(c) \geq f(x)$ for all x in the domain of f. Similarly, a function f has an **absolute (or global) minimum** at $x = c$ provided $f(c) \leq f(x)$ for all x in the domain of f.

Now is a good time to mention:
The Extreme Value Theorem: If f is continuous on the closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ for some c, d in the interval $[a, b]$.

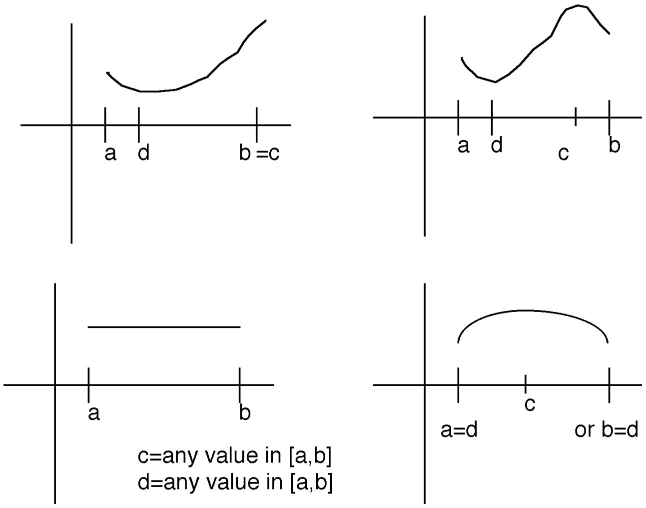

Figure 4.2: The extreme value theorem in action.

Example: Find the maxs and mins of $f(x) = x$ over all of the real numbers.

This has no answer. The extreme value theorem doesn't apply because we haven't restricted the function to a closed interval.

A function f has an **local maximum** at $x = c$ provided $f(c) \geq f(x)$ for all x close to c. Similarly, a function f has an **local minimum** at $x = c$ provided $f(c) \leq f(x)$ for all x close to c. Now we come to a very, very

nice theorem: **Fermat's Theorem** If $f(x)$ has a local maximum or a local minimum at $x = c$, and if $f'(c)$ exists, then $f'(c) = 0$. (see figure 4.1).

We can prove this one, and in the process review the definition of $f'(c)$. What a deal! Let's start the proof. We will only prove the case where $f(x)$ has a local min at $x = c$. The proof in the case of f having a max is similar:

Since $f(x)$ has a local min at $x = c$, we know that $f(x) - f(c) \geq 0$ for all x close to c. By definition,

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

Now, since this limit exists, it must equal the left hand limit and the right hand limit. Thus:

$$f'(c) = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} = \frac{+}{+} \geq 0$$

The + in the numerator indicates that $f(x) - f(c) \geq 0$ for values of x close to c, and the + in the denominator indicates that $x - c > 0$ because x is approaching c from the right. It follows that $f'(c) \geq 0$.

It must also be true that $f'(c)$ equals the left hand limit:

$$f'(c) = \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} = \frac{+}{-} \geq 0$$

Once again the + in the numerator indicates that $f(x) - f(c) \geq 0$ for values of x close to c, and the $-$ in the denominator indicates that $x - c < 0$ because x is approaching c from the left. It follows that $f'(c) \leq 0$. Since $0 \leq f'(c) \leq 0$, it must be true that $f'(c) = 0$. This concludes the proof of Fermat's Theorem.

There is one more type of number at which a max or minimum can occur, that is a value c at which $f'(c)$ does not exist. This inspires the definition of a **critical number** which is a number c at which $f'(c)$ is 0 or does not exist.

We now have the following theorem:

Theorem If $f(x)$ has a local maximum or local minimum at $x = c$, then either $f'(c) = 0$ or $f'(c)$ does not exist.

Finally, we discuss how to find global maxs and mins for a function defined over a closed interval.

Locating global maxs and mins: To find the absolute maximum and minimum values of a continuous function $f(x)$ over a closed interval $[a, b]$, do the following:

1. Find interior points that are critical points. Find out the value of $f(x)$ at these points.
2. Find the value of $f(x)$ at the endpoints.

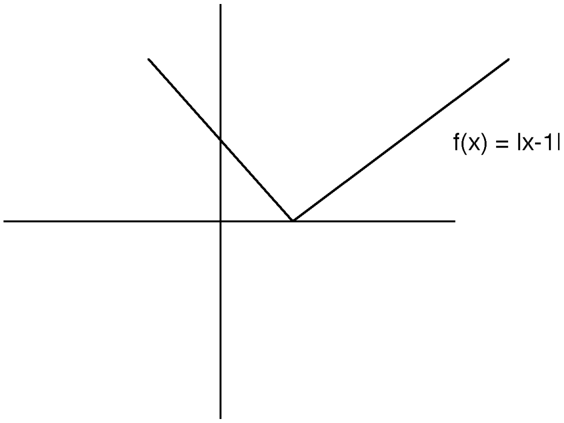

Figure 4.3: The graph of $y = |x-1|$.

3. Compare all of the values you found in the first two steps. The largest value is the absolute max, the smallest is the absolute min.

Example: Find the absolute maxs and mins of $f(x) = x^2 + 1$ given x in the interval $[-1, 2]$.

$f'(x) = 2x$. So $f'(x)$ is always defined, but $f'(x) = 2x = 0$ when $x = 0$. Check $f(0) = 1$. Keep this in mind....

Now we check endpoints. $f(-1) = 2$, and $f(2) = 5$. Now we have it. The absolute max is $f(2) = 5$. The absolute min is $f(0) = 1$.

Example: Find the absolute max and min for $f(x) = |x-1|$ on the interval $[-1, 3]$.

We'll spare an algebraic computation. Looking at the graph (see figure 4.3), we see that $f'(x)$ is never zero, but that it is undefined at $x = 1$. Check $f(1) = 0$. Now check endpoints: $f(-1) = |-1-1| = 2$, and $f(3) = |3-1| = 2$. So the absolute min is $f(1) = 0$, and the absolute max is $f(-1) = f(3) = 2$.

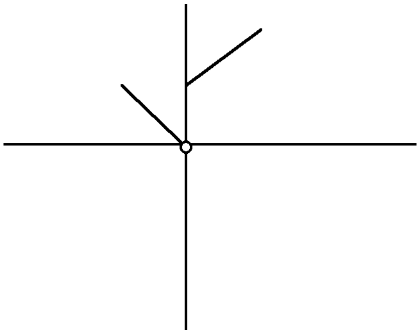

Figure 4.4: This function is not continuous and it doesn't have a global minimum value over the closed interval $[-1,1]$.

Example: Find the maxs and mins of $f(x)=\sqrt{4-x^2}$ for x in the interval $[-2,2]$.

$f'(x)=\dfrac{1}{2\sqrt{4-x^2}}(-2x)=\dfrac{-x}{\sqrt{4-x^2}}$. This is zero when the numerator is 0, namely when $x=0$. It is undefined when the denominator is zero, namely when $x=\pm 2$. Let's check: $f(0)=2$, $f(-2)=0=f(2)$. So our global max is $f(0)=2$, and our global min is $f(2)=0=f(-2)$.

Example:

$$f(x)=\begin{cases} x+1 & \text{if } x\geq 0 \\ -x & \text{if } x<0 \end{cases}$$

Find the global minimum of $f(x)$ for x in the interval $[-1,1]$.

This function has no minimum! How can that be? It isn't continuous. See figure 4.4.

4.2 The Mean Value Theorem

The Mean Value Theorem is an important piece of theory. It has some great consequences that we will use throughout our study of calculus. First, we start with Rolle's Theorem, which is a special case of the Mean Value Theorem.

4.2.1 Rolle's Theorem

Rolle's Theorem: Let f be a function satisfying:

1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b).
3. $f(a) = f(b)$

Then there is a c such that $a < c < b$ and $f'(c) = 0$.

Probably the best way to get a grasp of Rolle's Theorem is to examine some pictures (see Figure 4.5).

It's not too much work to prove Rolle's Theorem carefully, so we will do so now:

We treat three cases.

Case 1: $f(x) = k$ is constant. In this case all values of c such that $a < c < b$ have the property that $f'(c) = 0$. This case is done.

Case 2: $f(x) > f(a)$ for some x in the interval (a, b). By the Extreme Value Theorem (see previous section), f has to take on a global maximum value for some value in $[a, b]$. Since $f(x) > f(a)$ and $f(x) > f(b)$ (because by assumption 3 of the theorem, $f(a) = f(b)$), that maximum value does not occur at $x = a$ and the maximum value does not occur at $x = b$. It follows that the global maximum occurs at some value c, where c lies in (a, b). But if f has a global maximum at $x = c$, then it also has a local maximum at $x = c$. Since f is differentiable at $x = c$ (by assumption 2), then by Fermat's Theorem, $f'(c) = 0$. We're done with case 2.

Case 3: $f(x) < f(a)$ for some x in the interval (a, b). The proof for this case is almost the same as the proof in case 2. Mainly you have to change maxs to mins.

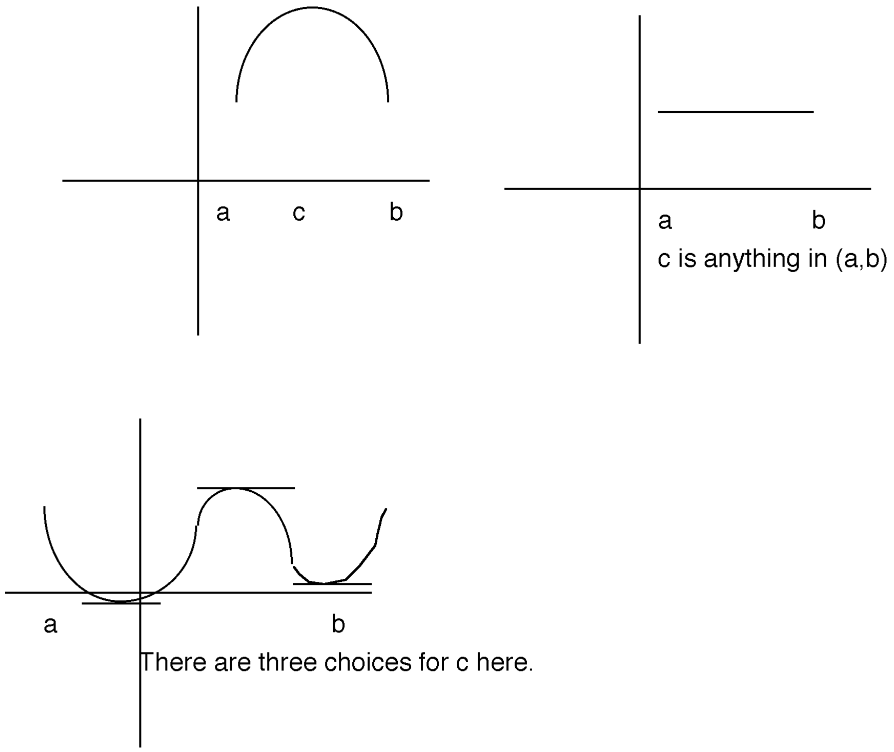

Figure 4.5: three graphs that illustrate Rolle's Theorem.

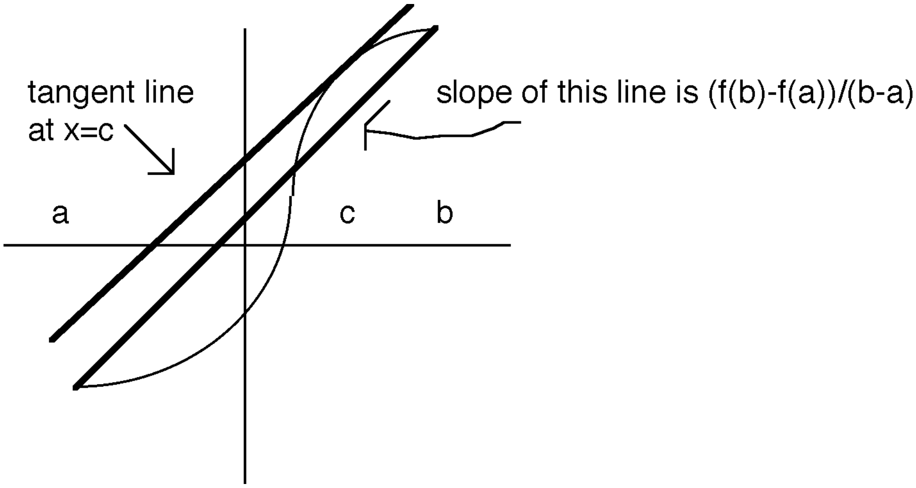

Figure 4.6: The instantaneous rate of change of f at some point c in (a, b) equals the average rate of change over the interval $[a, b]$.

4.2.2 The Mean Value Theorem

Now we'll state the Mean Value Theorem.

The Mean Value Theorem: Let f be a function that satisfies:

1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b).

Then there is a number c in (a, b) such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$. In other words, $f(b) - f(a) = f'(c)(b - a)$.

Another way to state the conclusion of the Mean Value Theorem is as follows: the instantaneous rate of change of f at some point c in (a, b) equals the average rate of change over the interval $[a, b]$. We won't prove this Theorem in these notes. The proof uses Rolle's Theorem. We'll concentrate on trying to understand what the theorem is trying to say. A good place to start is with a picture or two.

We now discuss three examples of applications of the Mean Value Theorem.

Example: Let $f(x) = x^2$, and let x lie in the interval $[0, 2]$. Find a value c in $(0, 2)$ at which $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

In this case, $a = 0$ and $b = 2$, so $f(b) - f(a) = f(2) - f(0) = 4$. So $\dfrac{f(b) - f(a)}{b - a} = 4/2 = 2$. We need to find a c at which $f'(c) = 2$. In other words, $2c = 2$, which implies that $c = 1$.

Example: Given that $f'(x) \leq 10$ and that $f(1) = 5$, what is the largest that $f(2)$ can possibly be? You may assume that f is a differentiable function.

We want to get information on $f(2)$. The MVT gives us the equation: $f'(c) = \dfrac{f(2) - f(1)}{2 - 1} = \dfrac{f(2) - 5}{1} = f(2) - 5$. Now, since $f'(x) \leq 10$, we know that $f'(c) \leq 10$. Putting this together, we get: $f(2) - 5 \leq 10$, so $f(2) \leq 15$. We may conclude that the largest $f(2)$ can possibly be under these conditions is 15.

Example: Given that you just travelled from Potsdam to Canton (10 miles) in 10 minutes, did your speedometer ever read 60 miles per hour? You should assume that the distance function is differentiable.

We'll let $f(t)$ stand for the distance travelled after t hours. We're given that $f(0) = 0$ and $f(1/6) = 10$. So the average rate of change in f over that $1/6$ hour is: $\dfrac{f(1/6) - f(0)}{(1/6) - 0} = \dfrac{10}{(1/6)} = 60$. So your average speed for the trip for the journey was 60 miles per hour. The Mean Value Theorem tells us that yes, your instantaneous speed had to have been 60 miles per hour for at least one instant during your trip. So, yes, your speedometer did read 60.