3.9 Related Rates

In many practical applications of calculus, different related quantities vary together. In terms of algebra, there will be a single equation involving two variables that both depend on another variable, say t. For example, if you drive from Potsdam to Canton,then the quantity $x =$ (distance from Potsdam to where you are), and $y =$ (distance from Canton to where you are) both vary with respect to time. In addition, since Potsdam and Canton are 13 miles apart, x and y are related by the equation $x + y = 13$. Taking $\frac{d}{dt}$ of both sides of this equation gives: $\frac{dx}{dt} + \frac{dy}{dt} = 0$, or $\frac{dx}{dt} = -\frac{dy}{dt}$.

Virtually all of the problems you will see in this section will be word problems. To get yourself in the right direction, you might want to follow these **steps for solving related rates problems:**

1. Draw a Picture and name all involved variables and constants. Assume all variables are differentiable functions of t.

2. Write down the given numerical information in terms of the symbols you chose in the previous step.

3. Write down what you have been asked to find. Usually it's a rate, expressed as a derivative.

4. Write an equation that relates the variables.

5. Take $\frac{d}{dt}$ of both sides of the equation you just wrote down.

6. **Plug in given numerical information last.** The only numerical information you can plug in early is the value of constants. Other than that, you must wait before you plug in, or else varying quantities will be treated as constants, and their derivatives will appear to be 0 when they are not.

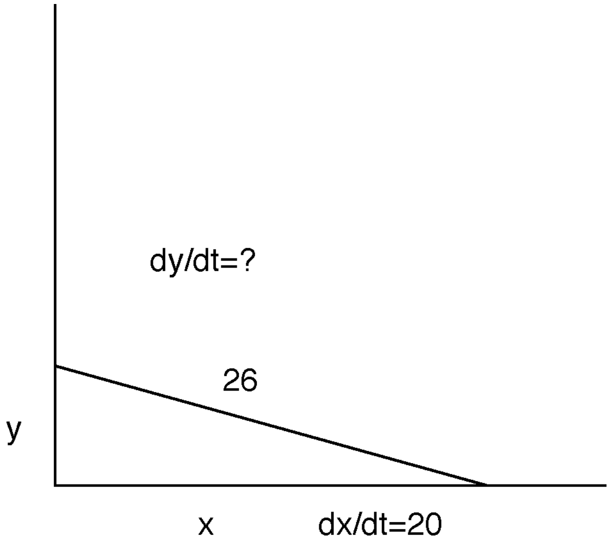

Figure 3.1: A ladder sliding down a wall.

Without any further ado, let's try an example.

Example: A 26 foot ladder resting on a horizontal floor is leaning against a vertical wall when its base starts to slide away from the wall. At the time the base is 24 feet from the wall, the base is moving at the rate of 20 feet per second. How fast is the top of the ladder moving down the wall at that point in time? How fast is the area of the triangle formed by the ladder, wall, and ground changing?

1. First, we draw a picture as in figure 3.1
2. In terms of the symbols chosen, we have $\frac{dx}{dt} = 20$ when $x = 24$. We'll use this information later.
3. We have been asked to find out how fast the top of the ladder is moving down the wall when $x = 24$. In other words, we have to find $\frac{dy}{dt}$ when $x = 24$. We have also been asked to find the rate of change of the area of the triangle. The area of the triangle is given by $(1/2)xy$. We need to find $\frac{d}{dt}[(1/2)xy]$.

4. As is common in these types of problems, the pythagorean theorem is involved. In this case, we have $x^2 + y^2 = 26^2$. So the values of x and y are related!

5. Now we differentiate the equation $x^2 + y^2 = 26^2$. The right hand side goes to zero, and we use the chain rule for the left hand side. We get:

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

6. Now, for the last step, we can plug in. We can use $x = 24$, $\frac{dx}{dt} = 20$. We also know what y is when $x = 24$. By the pythagorean theorem, $24^2 + y^2 = 26^2$. Using some algebra, we get $y = 10$. So, we can plug in $y = 10$ too. We get the following equation:

$$2(24)(20) + 2(10)\left(\frac{dy}{dt}\right) = 0$$

   In other words: $960 + 20\frac{dy}{dt} = 0$ Thus: $\frac{dy}{dt} = -48$. So the top of the ladder is falling at a rate of 48 feet per second. The negative sign indicates that the value of y is decreasing.

   We still have to find out the rate of change of the area of the triangle. Take $\frac{d}{dt}(1/2)xy$. Using the product rule, we get: $(1/2)(xy' + yx')$. Now plug in (always plug in last!) $= (1/2)(24(-48) + 10(20)) = 476$. Thus when $x = 24$ area is going down at a rate of -476 square feet per second.

Let's try another example:

Example: A light is at the top of a 16ft. pole. A person 5 ft. tall walks away from the pole at 4 feet per second. At what rate is the tip of their shadow moving when the person is 18ft. from the pole? At what rate is the length of their shadow increasing at that same instant?

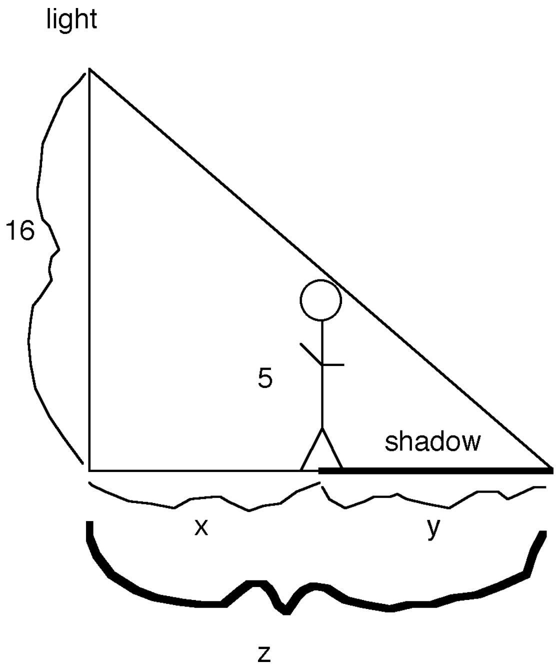

Figure 3.2: How fast is the tip of the shadow moving? How fast is the length of the shadow changing? The length of the shadow is represented by y. The position of the tip of the shadow is determined by z.

1. As usual, a good picture is essential. See figure 3.2.

2. We're given that $\frac{dx}{dt} = 4$ when $x = 18$.

3. We have been asked to find $\frac{dy}{dt}$ and $\frac{dz}{dt}$.

4. We can get an equation from the similar triangles involved. In particular, we get a ratio: $\frac{y}{5} = \frac{x+y}{16}$. Cross multiply to get $16y = 5x + 5y$. Then simplify to get $11y - 5x = 0$.

5. Now take d/dt: you get $11\frac{dy}{dt} - 5\frac{dx}{dt} = 0$.

6. We know that $\dfrac{dx}{dt} = 4$, so we can plug this is to and solve for $\dfrac{dy}{dt}$. We get $\dfrac{dy}{dt} = 20/11$. This tells us the length of the shadow is changing at 20/11 feet per second. We didn't even have to use $x = 18$.

   The tip of the shadow is determined by $z = x + y$. We can differentiate this expression with respect to t to get: $\dfrac{dz}{dt} = \dfrac{dx}{dt} + \dfrac{dy}{dt}$. At this instant we are interested in, $\dfrac{dx}{dt} = 4$, and $\dfrac{dy}{dt} = 20/11$. Combining we get: $\dfrac{dz}{dt} = 4 + 20/11 = 64/11$. So, the tip of the shadow is moving at 64/11 feet per second.

3.10 Linear Approximations and Differentials

Here's the main point of this section: given the graph of a function $y = f(x)$ that is differentiable, the tangent line to the graph at $x = a$ is close to the graph of $y = f(x)$ for values of x close to a. This fact comes up over and over again in calculus and in differential equations.

Since the tangent line is so important, let's remind ourselves how to get it's equation. We'll use the point-slope form of a line, and our line goes through $(a, f(a))$ with slope $f'(a)$:

$$\begin{aligned} y - f(a) &= f'(a)(x - a) \\ y &= f'(a)(x - a) + f(a) \end{aligned}$$

This is such a nice equation, that we call the function $L(x) = f'(a)(x - a) + f(a)$ the **linearization** of f at a. This is because the graph of $L(x)$ is a line that is close to $y = f(x)$ for values of x close to a.

Example Let $f(x) = \sin x$. Find the linearization of f at $x = 0$. Use $L(x)$ to approximate $\sin(.01)$, $\sin(1)$ and $\sin(.001)$.

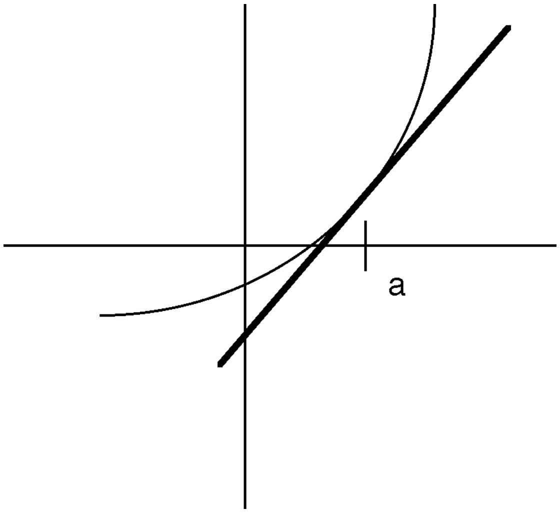

Figure 3.3: As you can see, the tangent line to the graph at $x = a$ is close to the graph of $y = f(x)$ for values of x close to a.

Of course, $f'(x) = \cos(x)$, so $f'(0) = 1$. Using the equation $L(x) = f'(0)(x - 0) + f(0)$, we have for this case, $L(x) = x$. We use this to approximate $\sin(.01) \approx .01$ (the TI-83 gives .0099998333), $\sin(.001) \approx .001$ (the TI-83 gives .0009999998333), and $\sin(1) \approx 1$ (the TI-83 gives .8414709848). So, $L(x)$ was pretty close for $x = .01$ and .001. For $x = 1$, you can see that the approximation wasn't that good. That is because for values of x not close to a, the graph of linearization of $f(x)$ at $x = a$ is very likely not to be close to the graph of $y = f(x)$.

In calculus II, you'll learn about Taylor Polynomials, which generalizes the idea of linearization.

3.10.1 Differentials

It is common to talk about linear approximations using the term differential. Without any further ado, let's define what a differential is. Given a differentiable function $y = f(x)$, we let dx be an independent variable (it can be given any real value, and it acts just like Δx), define $dy = f'(x)dx$. The variable dy (the **differential** of y) is dependent. It depends on the value of $f'(x)$ and on the chosen value of dx. In fact, dy stands for the amount of

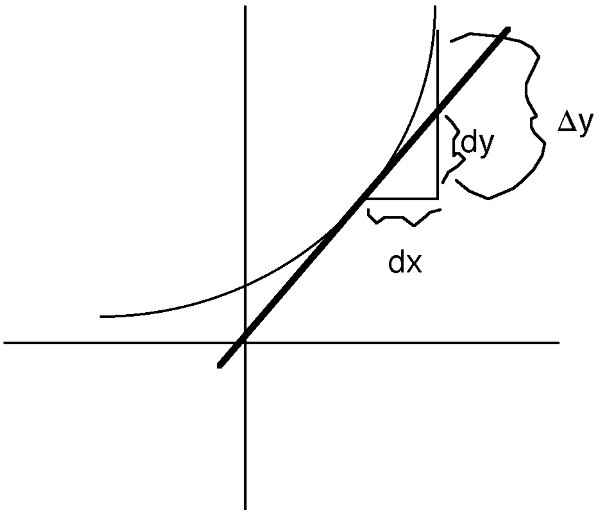

Figure 3.4: For small values of $\Delta x = dx$, we have that $dy \approx \Delta y$. Want to make them closer? Make dx smaller.

change in y along the tangent line, as x moves a distance of dx. The variable Δy represents the amount of change in y along the graph of $y = f(x)$, given a change in $x = \Delta x$. Since the tangent line at $x = a$ is really close to the graph of the function at $x = a$, for small values of $\Delta x = dx$, we have that $dy \approx \Delta y$.

Example: Let $y = f(x) = x^2 + 1$. Find Δy and dy if $\Delta x = dx = .1$ at $x = 2$.

Start with $dy = f'(x)dx = (2x)dx$. If $x = 2$ and $dx = .1$, then $dy = 4(.1) = .4$. That was so easy to compute!

To find Δy, use $f(2.1) - f(2) = 4.41 - .4 = .41$. That was a little more difficult to compute than dy. Note how close dy and Δy were.

Example: The edge of a square was found to be 15 inches with a possible error of ±.1 inches. Use differentials to estimate maximum error in computing the area of the square.

$A = x^2$, so $dA = 2xdx$. We're given that $dx = \pm.1$, and $x = 15$, so $dA = \pm 30(\pm.1) = \pm 3$ square inches. This is the total error. The **relative error** is the error/total area. In this case the relative error is $\frac{\pm 3}{225}$.

3.10.2 Summary

For each value of x close to a, the tangent line at $x = a$ is a good approximation of the differentiable function $y = f(x)$. The differential of y, $dy = f'(x)dx$. The variable dy represents change in y along the tangent line. For small $\Delta x = dx$, $\Delta y \approx dy$. Often dy is easier to compute than Δy. The idea of using the tangent line to approximate a function appears throughout calculus and differential equations.