3.6 The Chain Rule

In this section, you will add to you bag of tricks for differentiating functions. Don't forget the derivative represents slope, and don't forget that $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

The Chain Rule: If f and g are both differentiable then the composite function $f \circ g$ is differentiable, and

$$(f \circ g)'(x) = f'(g(x))(g'(x))$$

In Leibniz notation, if $y = f(u)$ and $u = g(x)$ are both differentiable, then $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$.

The idea behind the proof is that if an airplane travels 5 times as fast as a car, and if a car travels 4 times as fast as a bicycle, then the plane goes $5 \circ 4$ times as fast as the bicycle. In more technical terms, if Δu is small, then $\frac{\Delta y}{\Delta u} \approx \frac{dy}{du}$, and if Δx is small, then $\frac{\Delta u}{\Delta x} \approx \frac{du}{dx}$. Multiplying together, we get that for small Δu and small Δx, $\frac{\Delta y}{\Delta u}\frac{\Delta u}{\Delta x} = \frac{\Delta y}{\Delta x} \approx \frac{dy}{du}\frac{du}{dx}$. If we take a limit as $\Delta x \to 0$, we get exact equality: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$.

This is nowhere near a proof, but hopefully it gives you some indication of why the chain rule works. In terms of computing derivatives, which is what you are going to use the chain rule to do, you have to remember that to find $\frac{d}{dx}(f(g(x)))$, you have to first differentiate the outer function, namely f, then multiply by the derivative of the inside function, namely g.

Example: $\frac{d}{dx}(\sin x)^3$

Here the outside function is u^3. What's u, you might ask? It's $\sin x$, I would reply, which is also the inside function. Let's differentiate:

$$\frac{d}{dx}(\sin x)^3 = 3(\sin x)^2(\frac{d}{dx}(\sin x)).$$

We haven't finished yet....

$= 3(\sin x)^2(\cos x)$. That's better!

Example: $\frac{d}{dx}(\sin(x^2))$.

This time x^2 is the inside function. The derivative is:

$(\cos(x^2))(2x)$.

Example: Differentiate $(2x+1)^{1999}$.

The derivative is $1999(2x+1)^{1998}(2) = 3998(2x+1)^{1998}$.

Example: This time we'll do an example that requires an application of the chain rule twice. Find $\frac{d}{dx}(\sin(x^2))^3$.

It's $3(\sin(x^2))^2(\frac{d}{dx}\sin(x^2))$. But we're not done yet.

We need to find the derivative of $\sin(x^2)$. It's $(\cos(x^2))(2x)$. Let's put all of this together:

$$\frac{d}{dx}(\sin(x^2))^3 = 3(\sin(x^2))^2(\cos(x^2))(2x).$$

No go find some examples for yourself and practice the chain rule. It's the only way to master it.

3.7 Implicit Differentiation

Suppose you were given the problem of computing the tangent line to the unit circle ($x^2 + y^2 = 1$ at the point $(1/\sqrt{2}, 1/\sqrt{2})$? You could solve for $y = \sqrt{1-x^2}$, then differentiate to find the slope of this line. We're going to use a shortcut; we'll find the derivative without first solving for y.

Here's how: start with $x^2 + y^2 = 1$, then differentiate both sides of the equation with respect to x. Since you are differentiating with respect to x, you have to treat y like a separate function. Here's what you get:

$$2x + 2y(\frac{dy}{dx}) = 0.$$

Here we had to include $\frac{dy}{dx}$ because we don't know what $\frac{dy}{dx}$ is. We could have put $\frac{dx}{dx}$ after the $2x$, but this would be foolish since it is of course 1. Now we have to go back to our equation and solve for $\frac{dy}{dx}$. It's just algebra at this point:

$$2y(\frac{dy}{dx}) = -2x$$

Now divide both sides by $2y$:

$$\frac{dy}{dx} = \frac{-2x}{2y} = \frac{-x}{y}.$$

Let's get back to finding the tangent line at $(1/\sqrt{2}, 1/\sqrt{2})$. Plug in $(1/\sqrt{2}, 1/\sqrt{2})$ for (x, y) in the expression for $\frac{dy}{dx}$. We get $\frac{dy}{dx} = \frac{-1/\sqrt{2}}{1/\sqrt{2}} = -1$. So the slope is -1. The line is thus $y - 1/\sqrt{2} = (-1)(x - 1/\sqrt{2})$. Think about the tangent line to the circle at $(1/\sqrt{2}, 1/\sqrt{2})$. Does it appear to have slope -1?

Example: Given $\sin(xy) = 10x^2 + 10x + 1$, find $\frac{dy}{dx}$.

This one would be difficult if not impossible to solve for y, so we go with the technique of implicit differentiation. $\cos(xy)(x\frac{dy}{dx} + (1)y) = 20x + 10$. Notice how we had to use both the product rule and implicit differentiation to find the derivative of the product (xy). Now we solve for $\frac{dy}{dx}$. We get:

$\cos(xy)(x\frac{dy}{dx}) + \cos(xy)(y) = 20x + 10$. This becomes:

$\cos(xy)(x\frac{dy}{dx}) = -\cos(xy)(y) + 20x + 10$

$$\text{Thus: } \frac{dy}{dx} = \frac{-\cos(xy)(y) + 20x + 10}{x\cos(xy)}.$$

Phew! Now go find some examples and practice, practice, practice!!!