3.3 Differentiation Rules

Here's $f'(a)$ again in its two different forms:

$$f'(a) = \lim_{x \to a} \frac{f(h) - f(a)}{x - a}$$

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

In the last section, we saw that we can simply replace a with x to get:

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

The derivative is such a popular concept that it has many different symbols, check out p. 140 of your text to see some of its many guises.

It is sometimes cumbersome to compute the derivative of a function directly from the definition from above, so we're going to develop some shortcuts that will allow us to compute derivatives of functions accurately and quickly. You should not forget, however, how the derivative is defined, nor should you forget the the derivative, $f'(a)$ represents the slope of the tangent line to $y = f(x)$ at $x = a$.

Here's the first rule, you should not forget it. It says that the derivative of a constant is 0:

$$\frac{d}{dx} c = 0$$

Here's why: Since the function involved is $f(x) = c$ for any x, we have that $f(x + h) = c$, and $f(x) = c$, so $f(x + h) - f(x) = c - c = 0$. So, getting back to the definition,

$$f'(x) = \lim_{h \to 0} \frac{0}{h} = \lim_{h \to 0} 0 = 0.$$

Ta-Da!

3.3.1 The Power Rule

Your text establishes this rule in two different ways. We'll discuss one way, and you can look up the other way if you are interested. Start with $f(x) = x^n$, where n is a positive integer. We want to compute $f'(x)$. We'll use the following equality: $x^n - a^n = (x-a)(x^{n-1} + ax^{n-2} + a^2x^{n-3} \ldots + a^{n-3}x^2 + a^{n-2}x + a^{n-1})$. Using this, we can now compute:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{x^n - a^n}{x - a}$$

$$= \lim_{x \to a} \frac{(x-a)(x^{n-1} + ax^{n-2} + a^2x^{n-3} \ldots + a^{n-3}x^2 + a^{n-2}x + a^{n-1})}{x - a}$$

$$= \lim_{x \to a}(x^{n-1} + ax^{n-2} + a^2x^{n-3} \ldots + a^{n-3}x^2 + a^{n-2}x + a^{n-1})$$

$$= a^{n-1} + a^{n-2}a + a^{n-3}a^2 + \ldots + aa^{n-2} + a^{n-1}$$

$= a^{n-1}$ plus itself n times.

$$= na^{n-1}.$$

Swapping a for x, we get the **Power Rule:** If n is a positive integer, then:

$$\frac{d}{dx}x^n = nx^{n-1}$$

Now you know, for <u>Example:</u> $\frac{d}{dx}x^6 = 6x^5$.

We'll put the next three rules together:

The Constant Multiple Rule: If c is a constant and if f is a differentiable function, then:
$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}f(x)$$

The Sum Rule: If f and g are both differentiable functions, then:
$$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

The Difference Rule: If f and g are both differentiable functions, then:
$$\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$$

Let's prove the Sum Rule here, the proof of the difference rule is similar:

$$\frac{d}{dx}[f(x)+g(x)] = \lim_{h\to 0}\frac{f(x+h)+g(x+h)-[f(x)+g(x)]}{h}$$

By rearranging, this is:

$$= \lim_{h\to 0}\frac{(f(x+h)-f(x))+(g(x+h)-g(x))}{h}$$

Using properties of fractions, we get: $= \lim_{h\to 0}\left(\left(\frac{f(x+h)-f(x)}{h}\right)+\left(\frac{g(x+h)-g(x)}{h}\right)\right)$

Now we apply limit law number 1 from p. 84 of your text (it's the sum law, which says that the limit of the sums is the sum of the limits, provided each limit exists. We know each limit exists because f and g are differentiable):

$$= \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)+\lim_{h\to 0}\left(\frac{g(x+h)-g(x)}{h}\right)$$

By definition, this sum is:

$= f'(x)+g'(x)$. This is what we wanted!

The rules just discussed are not too surprising. The next two rules are perhaps a bit surprising:

The Product Rule: If f and g are differentiable, then:
$(fg)'(x) = f'(x)g(x)+g'(x)f(x)$

The Quotient Rule: If f and g are differentiable, then:
$(f/g)'(x) = \dfrac{f'(x)g(x)-g'(x)f(x)}{(g(x))^2}$

Before we prove the product rule, let's consider the function $f(x)=x^2$. We just computed $f'(x)$ in the previous section, it's $2x$. If you try to write $\frac{d}{dx}x^2 = \frac{d}{dx}x\frac{d}{dx}x = (1)(1) = 1$, then you know you've done something wrong. You should have used the product rule. I told you it was surprising (but true). Let's try again: $\frac{d}{dx}x^2 = \frac{d}{dx}(x)x + x\frac{d}{dx}x = (1)x + x(1) = 2x$. Ah, that's better! The product rule rules!

Now we'll prove the product rule. Of course, we have to use the definition of derivative to prove it:

$$\frac{d}{dx}(fg)(x) = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h}$$

What did we do there? We added and subtracted $f(x+h)g(x)$ to the numerator, which is just like adding 0 to the numerator, which changes nothing, except it will let us regroup terms here:

$$= \lim_{h\to 0} \frac{f(x+h)[g(x+h) - g(x)] + g(x)[f(x+h) - f(x)]}{h}$$

Now we'll use the sum rule for limits:

$$= \lim_{h\to 0} f(x+h)\left(\frac{g(x+h) - g(x)}{h}\right) + \lim_{h\to 0} g(x)\left(\frac{f(x+h) - f(x)}{h}\right)$$

By the product rule of limits, this is:

$$= \lim_{h\to 0} f(x+h) \lim_{h\to 0}\left(\frac{g(x+h) - g(x)}{h}\right) + \lim_{h\to 0} g(x) \lim_{h\to 0}\left(\frac{f(x+h) - f(x)}{h}\right)$$

Now, since f is differentiable, it is also continuous, so we can find the first limit by plugging in! The other limits are also easily recognized to give the following:

$$= f(x)g'(x) + f'(x)g(x).$$

Done!

Example: Find $\frac{d}{dx}(x^3+1)(x^{99}+12)$.

It's $3x^2(x^{99}+12) + x^3(99x^{98})$.

We won't prove the quotient rule here. It can be found in your text. After section 3.6, we'll be able to prove it using the chain rule and the product rule. For now, we'll do an example:

Example: Find $\frac{d}{dx}\left[\frac{x^2+7}{x^5}\right]$.

It's $\frac{(2x)(x^5)-(5x^4)(x^2+7)}{x^{10}}$

Now that you know all of these shortcuts, don't forget where they all come from, namely the definition of the derivative:
$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$.

And don't forget that the derivative of a function at a point represents the slope of the tangent line at that point.