

3.1 Derivatives

The slope of the tangent line to $y = f(x)$ at $x = a$ is such an important topic in calculus, that it gets its own special name, the **derivative** of $y = f(x)$ at $x = a$. We denote it by $f'(a)$. As in the previous section, we have the following:

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

and equivalently,

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

So, remember this: The **derivative** of $y = f(x)$ at $x = a$ is the **slope** of the tangent line to $y = f(x)$ at $x = a$. Every morning, before you brush your teeth, you should repeat the mantra: “**derivative gives slope, derivative gives slope, derivative gives slope.**” This should help you remember what the derivative represents in the distant future, like when you are taking calc II.

The derivative also represents the **instantaneous rate of change** of $y = f(x)$ at $x = a$. This is because the derivative is a limit of quotients of the form $\frac{\Delta y}{\Delta x}$, which gives the rate of change of y with respect to x . If we let the change in x go to zero (which is what happens when we compute the derivative), then we get the instantaneous rate of change.

3.2 The Derivative as a Function

It shouldn't be surprising that we can write $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$, provided the limit exists at that particular value of x . Let's use this to compute the derivative of some specific functions.

Example: Use the definition to compute the derivative of $f(x) = x^2$.

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(x+h)^2 - x^2}{h} \end{aligned}$$

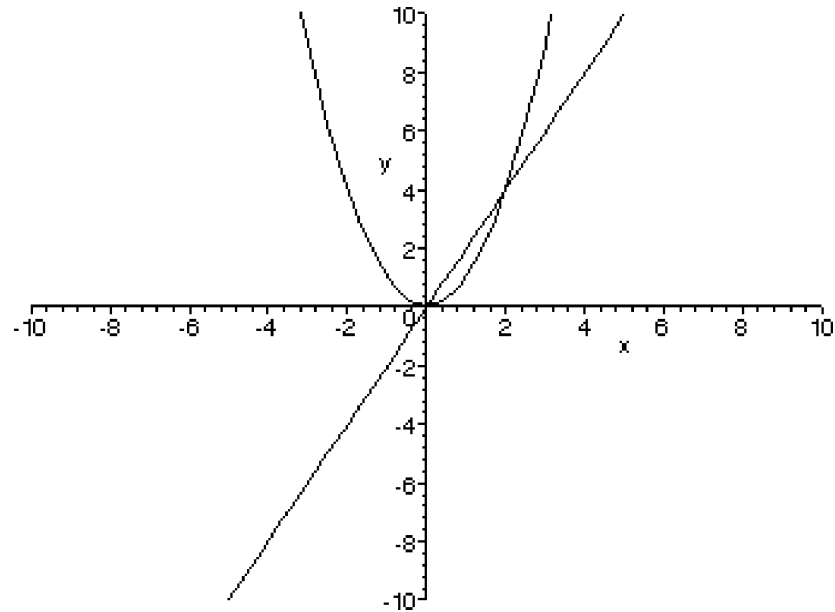


Figure 3.1: The graph of $y = x^2$ together with its derivative $y = 2x$.

$$= \lim_{h \rightarrow 0} \frac{x^2 + 2xh + h^2 - x^2}{h}$$

$$\text{Cancel } x^2\text{'s to get: } = \lim_{h \rightarrow 0} \frac{2xh + h^2}{h}$$

Now cancel h 's:

$$= \lim_{h \rightarrow 0} (2x + h)$$

$$= 2x.$$

This means that the derivative of $y = f(x)$ at a given value of x is $2x$. So, at $x = 2$, the derivative is 4. Let's try to interpret this geometrically.

Never, ever forget that the derivative is a slope giving machine. So, at $x = 1$, if we want to find the slope of $y = x^2$, we plug $x = 1$ into the derivative function $f'(1) = 2(1) = 2$. So, at $x = 1$, the function $y = x^2$ has slope 2. This should make sense, since the graph does seem to be rising as you look from left to right (at least near the point $x = 1$), just like a line with positive slope does. What is the slope at $x = 3$ you ask? If you look at the derivative function, you should notice that the derivative is positive for values of x near 3 (its graph lies above the x -axis). To find the exact slope, just plug into f' , it's $2(3) = 6$, which is indeed positive. This also indicates that the graph

is steeper at $x = 3$ (slope 6) than it was at $x = 2$ (slope 2). Does it look steeper to you? It should.

Finally, let's examine the slope at $x = -2$. Again, we use the derivative. If you look at the graph of the derivative at $x = 2$, you should notice that the derivative is negative (it lies below the x -axis) around where $x = 2$. This indicates that $y = x^2$ should have a negative slope. Plugging in, we find that the slope is exactly $2(-2) = -4$, which is indeed negative. Notice how for values of x near -2 the graph of $y = x^2$ seems to be decreasing as you move from left to right, just like a line with negative slope does.

You may have noticed that the function $f'(x) = 2x$ has slope 2. Does this mean $y = x^2$ has a slope of 2 everywhere? The answer is NO. To find the slope of $y = x^2$ you have to plug in your x value into the derivative function. If you want to get technical, the slope of the derivative function is the SECOND derivative of the function, which is quite different than the derivative of the function.

Example: Use the definition of derivative to compute $\frac{d}{dx}\sqrt{x}$.

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \\ &= \lim_{h \rightarrow 0} \left(\frac{\sqrt{x+h} - \sqrt{x}}{h} \right) \left(\frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \right). \end{aligned}$$

(We just multiplied by a fancy form of 1.)

$$\begin{aligned} &= \lim_{h \rightarrow 0} \frac{x+h-x}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h \rightarrow 0} \frac{h}{h(\sqrt{x+h} + \sqrt{x})} \\ &= \lim_{h \rightarrow 0} \frac{1}{(\sqrt{x+h} + \sqrt{x})} \\ &= \frac{1}{2\sqrt{x}}. \end{aligned}$$

Looking of the graphs of $f(x) = \sqrt{x}$, and $f'(x) = \frac{1}{2\sqrt{x}}$, we see that $y = f(x)$ is increasing, and that $y = f'(x)$ is always positive (above the x -axis).

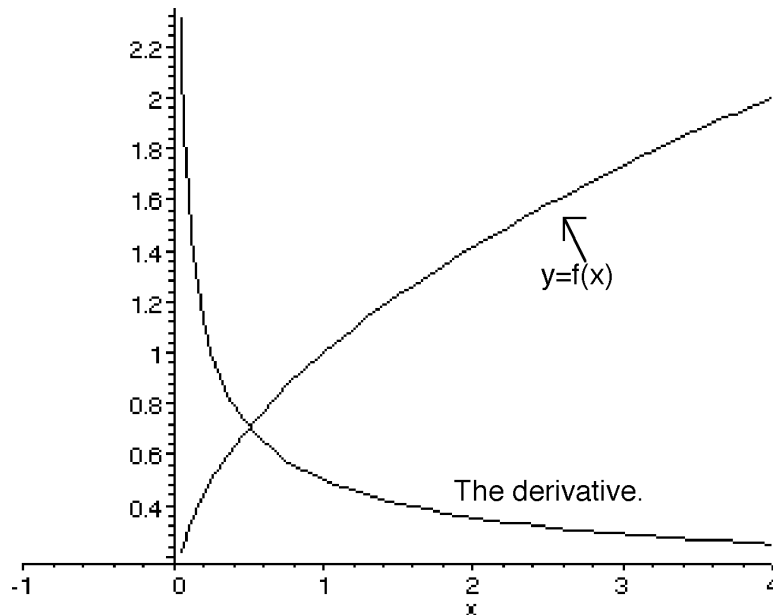


Figure 3.2: The graph of $y = \sqrt{x}$ and its derivative function.

We say a function f is **differentiable** at a if $f'(a)$ exists. We have the following nice theorem:

Theorem: If f is differentiable at a , then f is continuous at a .

Proof: Assume $f'(a)$ exists. We need to show that $\lim_{x \rightarrow a} f(x) = f(a)$. We'll do this by manipulating the expression $f(x) - f(a)$. Start with $f(x) - f(a)$ and multiply by $\frac{x-a}{x-a}$ (a fancy way to multiply by 1.) You get that $f(x) - f(a) = \frac{f(x) - f(a)}{x-a}(x-a)$ provided $x \neq a$.

We thus have $\lim_{x \rightarrow a} (f(x) - f(a)) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x-a} (x-a)$.

By the product law for limits (from chapter 2.3), this is:

$$\lim_{x \rightarrow a} \frac{f(x) - f(a)}{x-a} \lim_{x \rightarrow a} (x-a)$$

Because f is differentiable at a , and because $(x-a)$ is a continuous function at $x = a$, this limit is more simply:

$$= (f'(a))(0) = 0.$$

So, we just showed that $\lim_{x \rightarrow a} (f(x) - f(a)) = 0$. By the difference law (from 2.3 again), we know that $\lim_{x \rightarrow a} (f(x) - f(a)) = \lim_{x \rightarrow a} f(x) - \lim_{x \rightarrow a} f(a)$.

Since $f(a)$ is just a constant, we know that $\lim_{x \rightarrow a} f(a) = f(a)$. Putting all of this together, we get $\lim_{x \rightarrow a} f(x) - \lim_{x \rightarrow a} f(a)$

$$= \lim_{x \rightarrow a} f(x) - f(a)$$

$$= 0$$

Thus $\lim_{x \rightarrow a} f(x) = f(a)$. In other words, $f(x)$ is continuous at $x = a$.

3.2.1 When a function fails to be differentiable at a point

From the preceding theorem, a function that is **not continuous** at $x = a$ will not be differentiable there either.

If the graph of $y = f(x)$ is continuous but if it has a sharp corner or a kink at $x = a$, $f(x)$ will not be differentiable at $x = a$. (In trying to compute $f'(a)$, the left and right limits will not match up.)

A last way that $f'(a)$ can fail to exist is if the tangent line to $y = f(x)$ happens to be vertical. In this case, the slope will be undefined, and thus of course the derivative will also be undefined.

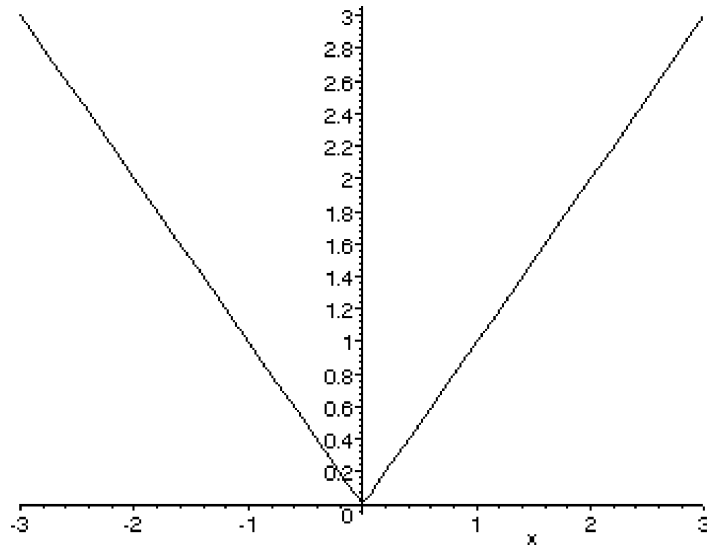


Figure 3.3: The graph of $y = |x|$ is not differentiable at $x = 0$. The function is continuous at $x = 0$

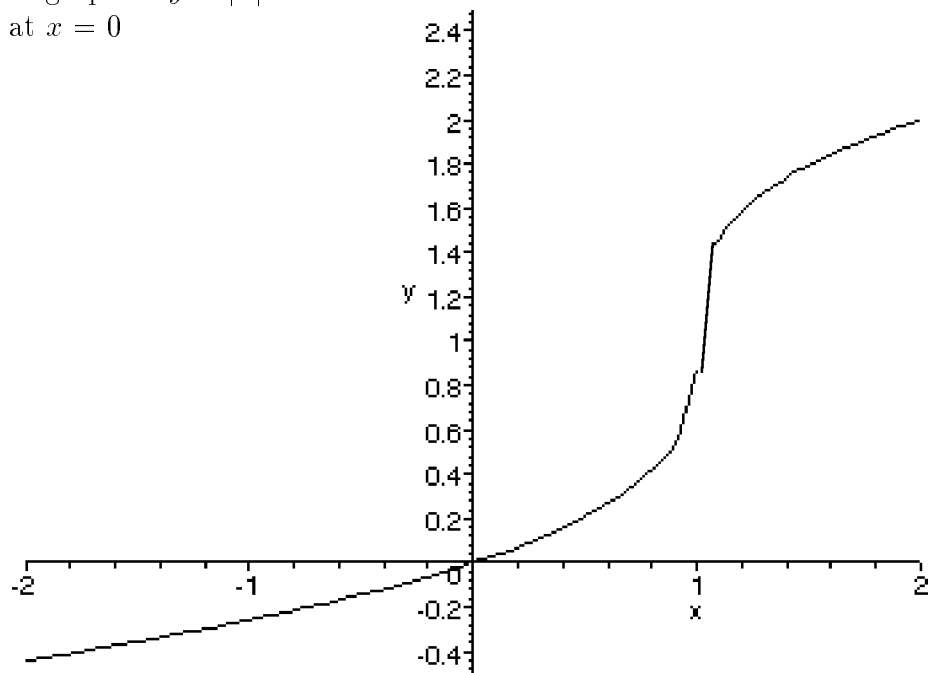


Figure 3.4: The graph of $y = \sqrt[3]{x-1} + 1$ is not differentiable at $x = 1$. The function is continuous at $x = 1$.

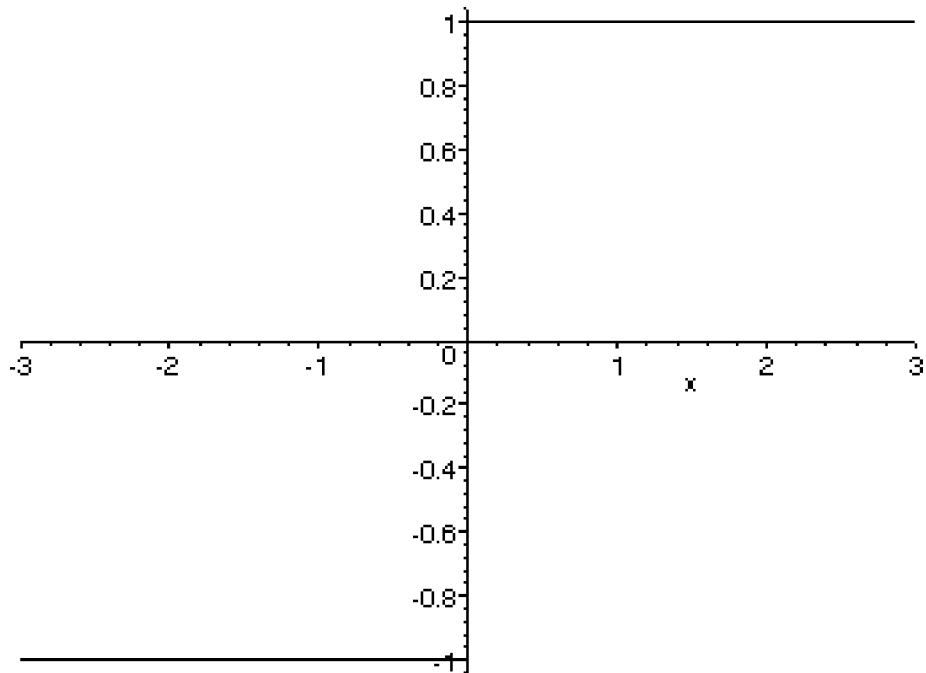


Figure 3.5: The graph of $y = \frac{x}{|x|}$ is neither differentiable nor continuous at $x = 0$.