2.5 Continuity

We have seen that for some functions, like polynomials, the limit as x approaches a can be found by "cheating" that is, by plugging in $x = a$ into the function. Functions with this nice property will be called **continuous**. Here is a definition:

Let f be a function defined on an interval I containing $x = a$. If

$$\lim_{x \to a} f(x) = f(a),$$

then f is **continuous** at $x = a$. Note that if a is a left or right endpoint of I, then a right-hand limit or left-hand limit replaces the preceding ordinary limit. To say that f is continuous on I means that f is continuous at each point of I (including endpoints of I, if any).

Note:

1. The **informal way** to recognize a continuous function on an interval I **is to check if the function's graph is unbroken** for x in I. This works as a rule of thumb for most functions you will encounter in this course.

2. A function may be continuous on one set but not continuous on another. **When you talk about a function being continuous, you must also specify where.**

3. As a rule, the elementary functions of calculus are continuous, where they are defined. For example, **polynomials, rational functions, and root functions are continuous at all points of their domain.** To prove this rigorously, one can use the limit laws from section 2.3. One can also show that the trigonometric functions are continuous at points in their domains. (if curious, see Stewart, p. 106 for an indication as to how this is done.) Note that some of the trig functions have "broken" graphs. It is after eliminating from the domain values where the breaks occur the function is continuous. For example, for $f(x) = \tan x$, we do not include $x = \pi/2$ in the domain, and the graph of $y = \tan x$ has an asymptote at $x = \pi/2$. The function $\tan x$ is not continuous at $x = \pi/2$, but we don't include this value in the domain.

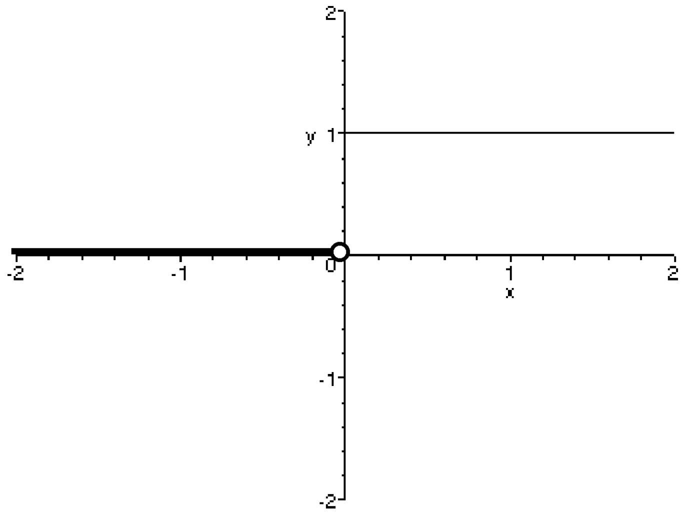

Figure 2.1: This function has a jump discontinuity at $t = 0$.

4. Finding limits of a continuous function is easy. You just plug in a for x. Now you can easily compute limits of trig functions and root functions, in addition to polynomials and rational functions.

   Example: Use the fact that $f(x) = \sin x$ is continuous to find $\lim_{x \to 0} f(x)$.

   Since $\sin x$ is continuous, $\lim_{x \to 0} \sin x = \sin(0) = 0$. Wasn't that easy?

5. There are three common types of discontinuities we want to highlight. Let's discuss them right now.

   (a) The function
   $$H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$$
   is discontinuous at $t = 0$. The type of discontinuity illustrated is called a **jump discontinuity**.

   (b) The functions
   $$f(t) = \begin{cases} \dfrac{1}{x^2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$
   is discontinuous at $t = 0$.

   We say that $f(x)$ has an **infinite** discontinuity at $x = 0$.

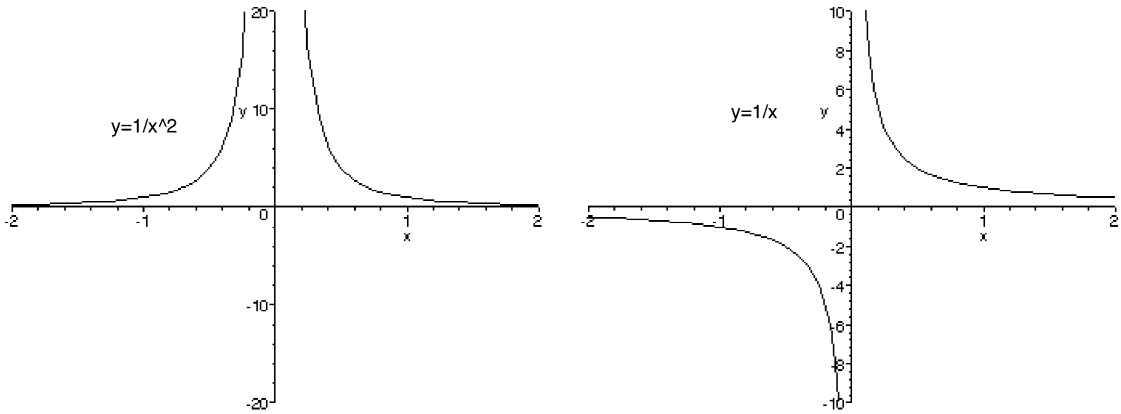

Figure 2.2: Both of these functions have infinite discontinuities at $x = 0$. (And 0 is not in the domain of either function.

(c) Finally, the function $f(x) = \dfrac{x^2 - 1}{x - 1}$ has a **removable discontinuity** at $x = 1$. It's graph looks just like the graph of $y = x + 1$, except there is a hole at $x = 1$. We could easily remove this discontinuity by redefining $f(1) = 2$, and letting $f(x)$ keep its values for $x \neq 1$.

6. So, there are three conditions that must be true for f to be continuous at $x = a$. The first is that the limit of $f(x)$ as x approaches a must exist, the second is that $f(a)$ must exist, and the third is that the limit and $f(a)$ must equal each other.

Example: At what point does the following function have a removable discontinuity? Redefine the function value at that point only to make the function continuous at all real numbers.

$$f(x) = \frac{x^2 + 4x + 4}{x + 2}$$

We see this is a rational function. The numerator can be factored into $(x + 2)(x + 2)$, canceling with the denominator, we get that $f(x) = x + 2$ for $x \neq -2$. This function thus has a removable discontinuity at $x = -2$. We redefine $f(-2) = \lim_{x \to -2} f(x) = 0$. So, if we redefine $f(-2) = 0$, we get a function that is continuous for all real numbers.

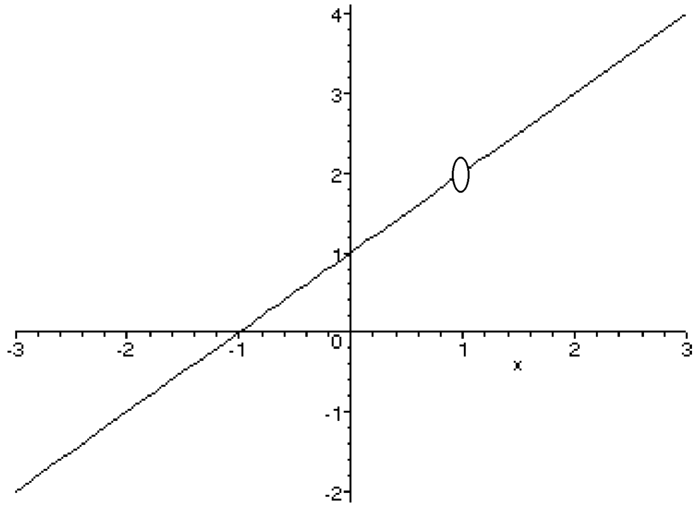

Figure 2.3: This function has a removable discontinuity at $x = 1$.

Example: This one is a little more tricky. Let $f(x) = \begin{cases} \mathrm{cx}^2 + 2 & \text{if } x < -2 \\ 2\mathrm{x} & \text{if } x \geq -2 \end{cases}$

What value of c makes f continuous at $x = -2$?

We need $\lim_{x \to -2} f(x) = f(-2)$. Viewing each piece separately, we see that:

$$\lim_{x \to -2^+} f(x) = -4$$

and that

$$\lim_{x \to -2^-} f(x) = 4c + 2$$

In order for the left and right limits to be equal, we need $4c + 2 = -4$, in other words, we need $c = -1.5$.

Finally, we can use the limit laws again to establish that the following combinations of continuous functions are continuous:

If f and g are continuous at $x = a$, and if c is a constant, then the following functions are continuous at $x = a$:

1. $f + g$
2. $f - g$

3. fg
4. cf
5. f/g, provided $g(a) \neq 0$.

One more technical fact: if g is continuous at $x = a$, and if f is continuous at $x = g(a)$, then $f \circ g$ is continuous at $x = a$. This says that the composition of continuous functions yields a continuous function. It can be proven directly from the limit laws (see Stewart, p. 110).

The Intermediate Value Theorem is important, and will be studied in a lab exercise.

2.6 Tangents and Velocities

2.6.1 Instantaneous Velocity

Suppose a car is traveling down the highway, and the the distance traveled after t hours is given by $s(t) = 20t^2$ miles. After 1 hour, the car has traveled 20 miles, after 2, it has traveled $s(2) = 20(2)^2 = 80$ miles, and after 3 hours, it has traveled $s(3) = 20(3)^2$ miles. From earlier work, you know how to compute the average velocity over the interval $[0, 1]$, it's $\frac{s(1) - s(0)}{1 - 0} = 20$. Over the interval $[1, 2]$, the average velocity is $\frac{s(2) - s(1)}{2 - 1} = \frac{80 - 20}{1} = 60$ miles per hour, and over the interval $[2, 3]$, the average speed is $\frac{s(3) - s(2)}{3 - 2} = \frac{180 - 80}{1} = 100$ miles per hour. The car is speeding up! Can we find the car's velocity at any instant?

If $s(t)$ gives the car's odometer reading at time t, the distance traveled between time t and time $t + h$ is $s(t + h) - s(t)$. The difficulty in computing the velocity at an instant is that the distance traveled and the time elapsed are both zero, so you cannot divide. The way to get around this is to divide by shorter and shorter period of length, h.

In general, we can define the **instantaneous velocity** of $y = s(t)$ at time $t = a$ to be the limit of the average velocities as the interval about a goes to 0:

$$v(a) = \lim_{h \to 0} \frac{s(a+h) - s(a)}{h}$$

In our car example, we can compute the velocity at the instant $x = 1$ via: $\lim_{h \to 0} \frac{s(1+h) - s(1)}{h} = \lim_{h \to 0} \frac{20(1+h)^2 - 20(1)^2}{h} = \lim_{h \to 0} \frac{20(1+2h+h^2 - 20}{h} = \lim_{h \to 0} \frac{40h + 20h^2}{h} = \lim_{h \to 0} 40 + 20h = 40$. Now we know that at $t = 1$, the car was traveling 40 miles per hour.

2.6.2 Tangent Lines

Consider the function $f(x) = 20x^2$. Let's try to find the tangent line to $y = f(x)$ at $x = 1$. We'll do this by considering a nearby point of the form $(x, f(x))$, where $x \neq 1$, and then we'll compute the slope of the secant line through the two points: $m = \frac{f(x) - f(1)}{x - 1}$. Then we let the point $(x, f(x))$ approach the first by letting x approach 1. More precisely, the slope of the tangent line to our function is given by $\lim_{x \to 1} \frac{f(x) - f(1)}{x - 1}$, provided the limit exists. In our example, this becomes: $\lim_{x \to 1} \frac{20x^2 - 20}{x - 1} = \lim_{x \to 1} \frac{20(x^2 - 1)}{x - 1} = \lim_{x \to 1} \frac{20(x-1)(x+1)}{x - 1} = \lim_{x \to 1} \frac{20(x+1)}{1}$, which is 40. This shouldn't surprise you, given that this function and the distance function from the previous section were the same, and you computed instantaneous velocity at $t = 1$ to be 40 miles per hour. The tangent line at $x = 1$ thus passes through the point $(1, 20)$ and has slope 40. Its equation is then: $y - 20 = 40(x - 1)$.

In general, the **tangent line** to the curve $y = f(x)$ at the point $(a, f(a))$ is the line through $(a, f(a))$ with slope

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

provided the limit exists.

There is an equivalent definition for the slope of the tangent line through $(a, f(a))$. It is given by $m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$

(Let $h = x - a$, as $h \to 0$, we have $x - a \to 0$, so $x \to a$.)

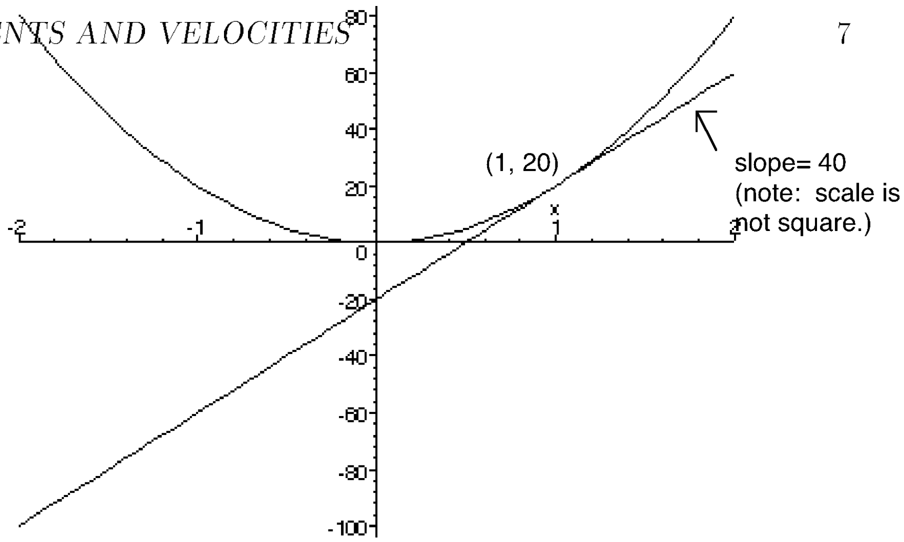

Figure 2.4: The tangent line to $y = 20x^2$ at $x = 1$.

Again, we see that if $y = f(t)$ represents distance traveled, the slope of the tangent line to $y = f(t)$ at $x = a$ gives the instantaneous velocity.

Example: Find the slope of the tangent line to $y = 2 - x^2$ at the point $(1, 1)$.

We'll use the second definition. $m = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{2 - (1+h)^2 - 1}{h} = \lim_{h \to 0} \frac{2 - (1 + 2h - h^2) - 1}{h} = \lim_{h \to 0} \frac{-2h - h^2}{h} = \lim_{h \to 0} -2 - h = -2$. So the slope is -2.

For this same example, the equation for the tangent line through $(1, 1)$ is given by $y - 1 = -2(x - 1)$, or more simply $y = -2x + 3$ (see figure 2.6).

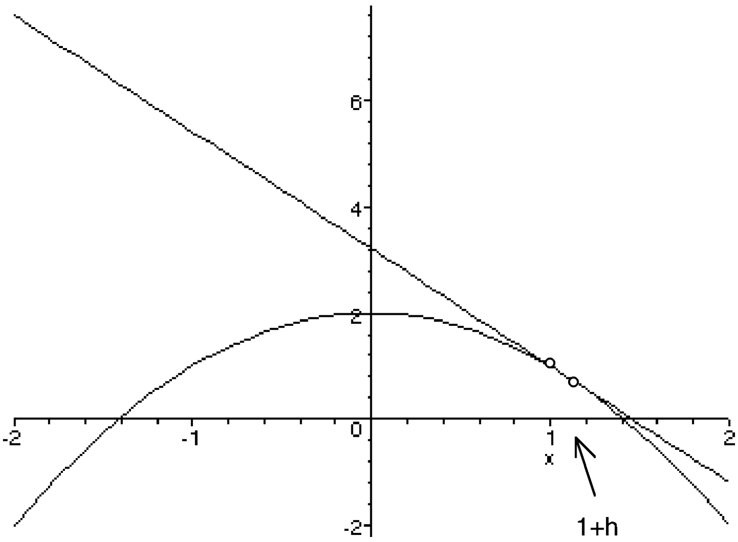

Figure 2.5: The secant line through $(1, f(1))$ and $(1+h, f(1+h))$. If we let h approach zero, this secant line will approach the tangent line at $(1, f(1))$.

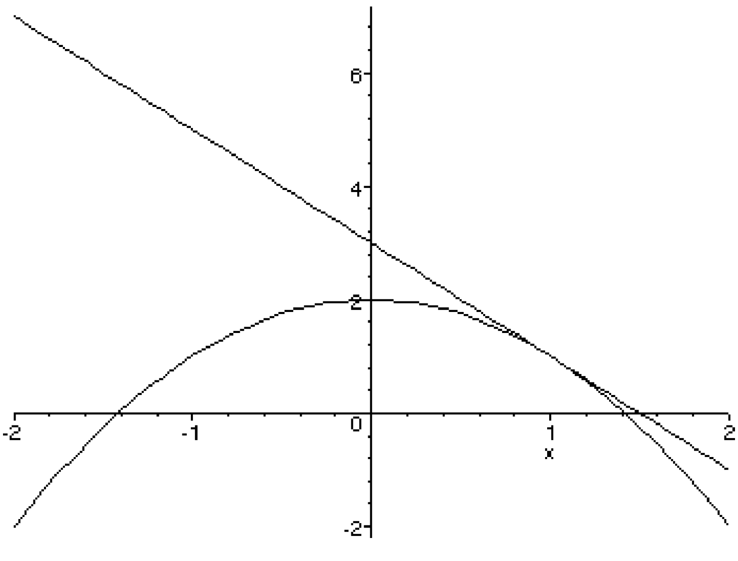

Figure 2.6: The tangent line to $y = 2 - x^2$ at $x = 1$.