2.3 Using Properties of Limits to Calculate Limits

In the previous section, you used tables and graphs to guess the values of certain limits. This chapter presents some rules you can use to calculate limits precisely. The rules can be proved using the formal definition (that you will learn about in 2.4). If you take advanced calculus (Math 451) you will probably see proofs of most of these rules (you can also look in appendix F of Stewart if you are curious). For now, you can accept them as laws. If you want to cut to the chase, you should skip the next bit until you come to the Direct Substitution Laws.

Limit Laws Suppose c is a constant, and that $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist. Then:

1. $\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$.
2. $\lim_{x\to a}[f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$.
3. $\lim_{x\to a}[cf(x)] = c\lim_{x\to a} f(x)$.
4. $\lim_{x\to a}[f(x)g(x)] = (\lim_{x\to a} f(x))(\lim_{x\to a} g(x))$.
5. $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$, provided $\lim_{x\to a} g(x) \neq 0$.
6. If m and n are integers, then $\lim_{x\to a}[f(x)]^{m/n} = [\lim_{x\to a} f(x)]^{m/n}$.

   We also need the following two facts, which are believable, and can be proved after we've discussed section 2.4.

7. $\lim_{x\to a} c = c$ and
8. $\lim_{x\to a} x = a$.

Here's a typical example of what you can do with these rules:

Example: Use the rules of limits to find $\lim\limits_{x\to 2} \dfrac{x^3+4x^2-3}{x^2+5}$. Justify each step.

By rule 7, $\lim\limits_{x\to 2} 5 = 5$. We'll use this later. By rule 8, $\lim\limits_{x\to 2} x = 2$. Now, by rule 6, we know that $\lim\limits_{x^2\to 2} x = 2^2 = 4$, and that $\lim\limits_{x^3\to 2} x = 2^3 = 8$.

Now let's try to break down the whole thing: By rule 5, $\lim\limits_{x\to 2} \dfrac{x^3+4x^2-3}{x^2+5} = \dfrac{\lim\limits_{x\to 2} x^3+4x^2-3}{\lim\limits_{x\to 2} x^2+5}$. This says we can compute our limit by finding the limits of the numerator and denominator separately. In particular, we will see that $\lim\limits_{x\to 2} x^2+5 \neq 0$. If the denominator had a limit of 0, we would have to use some other way to compute the limit of the fraction, because we can never, ever divide by 0.

Let's take on the numerator. We can use rules 1 and 2 to break down $\lim\limits_{x\to 2} x^3+4x^2-3 = \lim\limits_{x\to 2} x^3 + \lim\limits_{x\to 2} 4x^2 - \lim\limits_{x\to 2} 3$. By rule 3, this is the same as $\lim\limits_{x\to 2} x^3 + 4\lim\limits_{x\to 2} x^2 - \lim\limits_{x\to 2} 3$. Now, by the computations done above, we get this to be $8+16-3=21$. This is our numerator limit.

Now let's work on the denominator. It's a little bit easier. We start by using rule 1: $\lim\limits_{x\to 2} x^2+5 = \lim\limits_{x\to 2} x^2 + \lim\limits_{x\to 2} 5$. By our computations at the beginning, and by rule 7, this is equal to $4+5=9$. This is the limit of our denominator. By rule 5, the limit of the function we started with at the very beginning is $21/9$.

This is the point where students ask if there is an easier way. Of course there is, but we had to show you the previous example, because mathematics is a subject in which everything you do must be justified carefully. Can you appreciate that? Perhaps you will when you take advanced calculus (it's really not that difficult a course, you go through a lot of proofs very slowly). Here is the shortcut.

2.3.1 Direct Substitution Laws

It can be proved using the limit laws (see Stewart, exercises 53 in 2.3) they state that **limits of polynomials can be found by substitution, and the**

limits of rational functions can be found by substitution provided the denominator does not have a limit of 0:

If f is a polynomial or rational function and a is in the domain of f, then $\lim_{x\to a} f(x) = f(a)$.

Example: Find $\lim_{x\to 2}(x^2 + 6x + 1)$.

Now this is really easy. Just plug in 2 for x. We get that the limit is $2^2 + 6(2) + 1 = 17$.

Another important technique is through **canceling a common factor of a quotient.** The next example illustrates this technique.

Example: Calculate: $\lim_{x\to 2} \frac{x^2 - 4x + 4}{x - 2}$.

Factor the numerator into $(x-2)(x+2)$. Then, for values of x not equal to 2, we have $\frac{x^2 - 4x + 4}{x - 2} = \frac{(x-2)(x+2)}{x-2} = x + 2$. Now we can use direct substitution: $\lim_{x\to 2} x^2 - 4x + 4x - 2 = \lim_{x\to 2} x + 2 = 2 + 2 = 4$.

CAVEAT: You cannot use direct substitution to find limits of functions that are not polynomials or rational functions.

2.3.2 The Squeeze Theorem

Students have a hard time with this one, although I don't think it's that bad. Here it is:

The Squeeze Theorem If $f(x) \leq g(x) \leq h(x)$ when x is near a (except possibly at a) and

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$$

then

$$\lim_{x\to a} g(x) = L$$

You see, $g(x)$ is between $f(x)$ and $h(x)$, so it is squeezed between them, and since $f(x)$ and $h(x)$ have the same limit at $x = a$, so must $g(x)$ have the same limit (see Figure! 2.1).

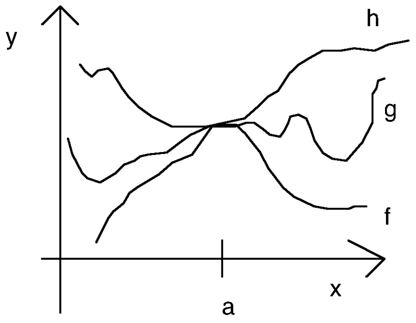

Figure 2.1: If $f(x) \leq g(x) \leq h(x)$ for all x near $x = a$, and $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$, thus $\lim_{x \to a} g(x) = L$.

As usual, the proof of this theorem can be found in Appendix 12 of Stewart, or you will probably see the proof in advanced calculus. **Chances are, in Calculus I, if you see a limit that can be established using the squeeze theorem, the function involved will have either** $\sin x$ **or** $\cos x$ **in it.** Here is a typical example, for another, see example 11 on p. 91 of Stewart.

Example: Show $\lim_{x \to 0} |x \cos(1/x)| = 0$.

We need to squeeze our given function between two other functions that have limiting value 0 as x approaches 0. OK, you should remember that for any value of $x \neq 0$, $-1 \leq \cos(1/x) \leq 1$. This is going to help us squeeze our function. It is similarly true that $|\cos(1/x)| \leq 1$. Here's our squeeze:

$$0 \leq |x \cos(1/x)| \leq |x(1)|$$

Of course $x(1) = x$. Now, $\lim_{x \to 0} |x| = 0$, and $\lim_{x \to 0} 0 = 0$, so we have successfully squeezed $|x \cos(1/x)|$ into having a limit of 0 as x approaches 0.

2.4 The Formal Definition of a Limit

The definition of limit given in section 2.2 is somewhat vague. For your convenience and enjoyment, we reprint it here:

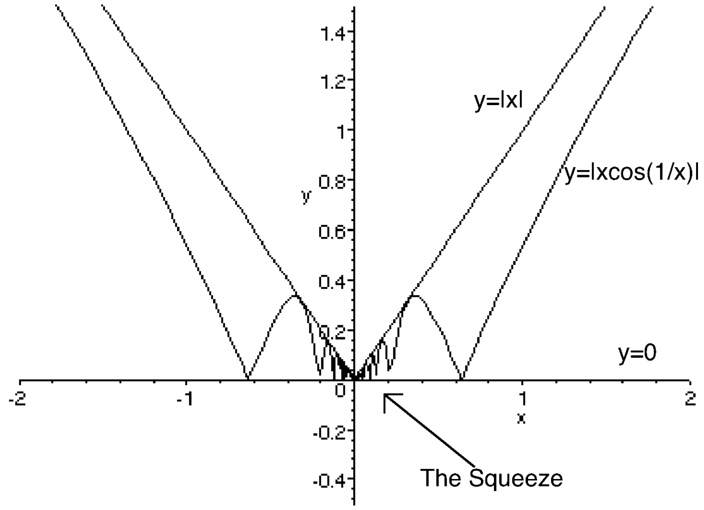

Figure 2.2: $\lim_{x\to 0} |x\cos(1/x)| = 0$

Let $f(x)$ be defined on an open interval about a, except possibly at a itself. We write

$$\lim_{x\to a} f(x) = L$$

and say "the **limit** of $f(x)$ as x approaches a is L" if we can make the values of $f(x)$ arbitrarily close to L (as close to L as we like) by choosing x to be sufficiently close to a, but not equal to a.

But, what do we mean by arbitrarily close? What is sufficient? We need a more formal and precise definition of a limit. The definition we are about to give bears the stamp of many mathematicians, particularly Cauchy and Weierstrass. It did not appear in its present form until the 1800's (calculus was first created in the 1600's). In other words, it took the brightest mathematicians 200 years to come up with this. Don't expect it to understand it after 5 minutes of study. In fact, you will study this definition again in advanced calculus. Without any further ado, here it is:

Definition: Let f be a function defined on some open interval that contains the number a, except possibly at a itself. We say the **limit of f(x) as x approaches a is L**, and we write:

$$\lim_{x\to a} f(x) = L$$

if for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that if $0 < |x - a| < \delta$ then $|f(x) - L| < \epsilon$.

Think of ϵ as specifying how close to L we want $f(x)$ to be, and δ as specifying how close to a we must make x in order to guarantee that $f(x)$ is within ϵ of L.

So, in the informal definition, they say we can make $f(x)$ "arbitrarily" close to L, provided x is sufficiently close to a. That arbitrariness is made specific by ϵ, and x being sufficiently close to a is made specific by δ. The reason for the 0 in the expression $0 < |x - a| < \delta$ is because we don't care about what happens when $x = a$. When $x = a$, $x - a = 0$, and so the $0 <$ eliminates this possibility

2.4.1 The ϵ, δ game.

From the informal chapter on limits, we can guess that $\lim_{x \to 4}(2x - 1) = 7$. If this is true, according to the formal definition, we can figure out how close to $a = 4$ we must hold the input x to be sure that $f(x)$ lies within, say, 1 unit of 7. In other words, given $\epsilon = 1$, we can find a δ such that if $0 < |x - 4| < \delta$, then $|f(x) - L| < \epsilon = 1$.

To find such δ, a good technique is to start with $|f(x) - L| < \epsilon$. Then do some algebraic manipulations to find δ.

In our example, $f(x) = 2x-1$, $L = 7$ and $\epsilon = 1$. So we start with $|2x-1-7| < 1$. In other words, $|2x-8| < 1$. We need to get this into a statement involving $|x - 4|$. If we divide both sides by 2, we get $|x - 4| < 1/2$. This leads us to our choice of $\delta = 1/2$, for if $0 < |x - 4| < 1/2$, then multiplying by 2 gives us that $0 < |2x - 8| < 1$ and thus that $|2x - 1 - 7| < 1$. So great, if we want to guarantee that $f(x)$ is within 1 unit of 7, all we have to do is make sure that $x$is within 1/2 units of 4. But can we guarantee that $f(x)$ is even closer to L?

Let's pick a smaller value for ϵ, say .5. Start with $|f(x) - L| < \epsilon$. Can we find an appropriate δ? We have that $f(x) = 2x - 1$, $L = 7$, and $\epsilon = .5$. So $|f(x) - L| < \epsilon$ becomes $|2x - 1 - 7| < .5$. This is more simply $|2x - 8| < .5$. Divide by 2 to get $|x - 4| < .25$. So, if we let $\delta = .25$, then if we have $0 < |x - 4| < .25$, then multiplying by 2 gives us that $0 < |2x - 8| < .5$, and

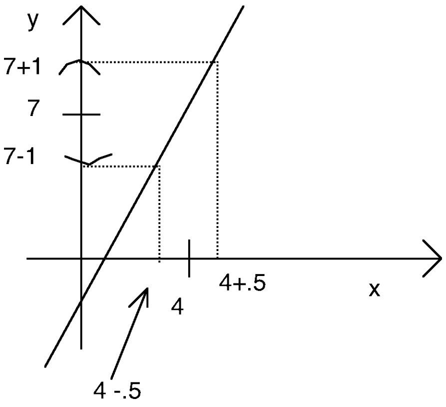

Figure 2.3: As long as x is within $\delta = .5$ of 4, $f(x)$ will be within 1 of 7.

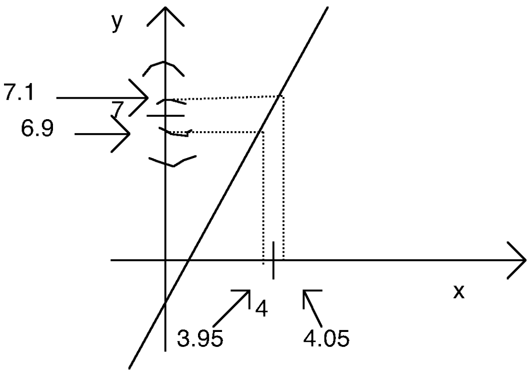

Figure 2.4: As long as x is within $\delta = .05$ of 4, $f(x)$ will be within .1 of 7.

this gives us that $|2x-1-7| < .5$, in other words that $|f(x)-L| < \epsilon$. Great, so if we want $f(x)$ within .5 of 7, we have to make x be within .25 of 4. But can we guarantee that $f(x)$ is even closer to L?

OK, let's try $\epsilon = .1$ this time. Start with $|f(x) - L| < \epsilon$. In other words, $|2x-8| < .1$. Divide both sides by 2 to get $|x-4| < .05$. This tells us to pick $\delta = .05$. If $0 < |x-4| < .05$, then multiplying by 2 yields that $|2x-8| < .1$, which is what we wanted to guarantee. So, we can guaranteed that $f(x)$ is within .1 of 7. But can we guarantee that $f(x)$ can be even closer to 7. We could try to pick an even smaller value of ϵ than we picked in the examples above, but we could then always ask if the process would work for an even smaller value. How can we get out of this seemingly endless process?

Leave ϵ as itself, and solve for δ as a function of ϵ. Let's try our example again. Start again with $|f(x)-L|<\epsilon$. As before, $f(x)=2x-1$, $L=7$, but this time $\epsilon=\epsilon$. We get $|2x-8|<\epsilon$. Divide by 2 to get $|x-4|<\epsilon/2$. So pick $\delta=\epsilon/2$. Then if $0<|x-4|<\epsilon/2$, then multiplying by 2 gives us that $|2x-8|<\epsilon$. In other words, given any $\epsilon>0$, we can find a $\delta>0$, namely $\delta=\epsilon/2$ such that if $0<|x-2|<\delta$, then $|f(x)-L|<\epsilon$. Since we can find an appropriate δ for any ϵ, we have rigorously proved that $\lim\limits_{x\to 4}(2x-1)=7$.

Example: Prove $\lim\limits_{x\to -4}(-3x+2)=14$

Start with $|f(x)-L|<\epsilon$. In this case, this is $|-3x+2-14|<\epsilon$, with is more simply $|-3x-12|<\epsilon$. We need find a δ for the inequality $0<|x-(-4)|<\delta$, which is $) < |x+4|<\delta$. Dividing both sides of $|-3x-12|<\epsilon$ by 3 gives us $|-x-4|<\epsilon/3$. Luckily for us, we can factor a (-1) out of $-x-4=(-1)(x+4)$. Now, since multiplying a number by -1 does not change its absolute value, our equation becomes:

$|(-1)(x+4)|<\epsilon/3$, which is more simply $|x+4|<\epsilon/3$. Can you tell what we need to pick for δ? If you guessed 10, you would be wrong (where did you get that from??), but if you guessed $\epsilon/3$, you would be correct. So, if $0<|x+4|<\epsilon/3$, then multiplying by 3 gives $|3x+12|<\epsilon$, and then multiplying inside the absolute value by (-1) gives $|-3x-12|<\epsilon$, which is the same as $|f(x)-L|<\epsilon$. Note that if you are writing up a problem like this for homework, you should circle your value for δ. That is the important number. Again, in this case, we have $\delta=\epsilon/3$. You may have been tempted above to set $\delta=\epsilon/(-3)$. This would not have worked, since it would have made δ negative. Be careful, and never pick δ to be negative. You could have picked δ to be *any* value greater than 0 and less than $\epsilon/3$. Can you see why?

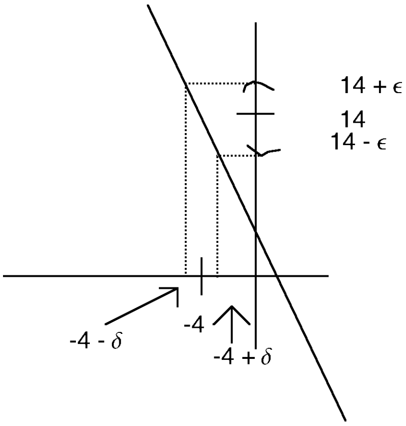

Figure 2.5: The limit of $-3x + 2$ as x approaches -3 is 14.